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# NONNEGATIVE MATRICES WITH PRESCRIBED SPECTRA

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A dissertation submitted in partial fulfillment
of the requirements for the degree of
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ABSTRACT

NONNEGATIVE MATRICES WITH PRESCRIBED SPECTRA

Patricia D. Egleston

Nonnegative matrices are rectangular arrays of nonnegative real numbers. Nonnegative matrices are prevalent in many areas of study. Statistics, Economics, and Chemistry are examples of disciplines that use nonnegative matrices. In this dissertation we examined and produced new results on nonnegative matrices with prescribed spectra. This problem is called the nonnegative inverse eigenvalue problem (NIEP). The NIEP asks which lists of $n$ complex numbers occur as the spectrum of an $n$ x $n$, entry-wise nonnegative matrix. A recent paper by Leal-Duarte and Johnson stated, "this problem has attracted considerable attention over 50 years and, despite many exciting partial results, remains quite unresolved." In this dissertation we solved the NIEP for certain lists of four complex numbers.

Since the general NIEP is difficult to resolve, two natural variations were considered. The real nonnegative inverse eigenvalue problem (RNIEP) asks which lists of $n$ real numbers occur as the spectrum of a nonnegative $n$ x $n$ matrix. The symmetric nonnegative inverse eigenvalue problem (SNIEP) asks which lists of $n$ real numbers

occur as the spectrum of a nonnegative $n$ x $n$ matrix. The symmetric nonnegative inverse eigenvalue problem (SNIEP) asks which lists of $n$ real numbers occur as the spectrum of a symmetric nonnegative $n$ x $n$ matrix. We resolved the SNIEP in most of the cases for lists of five real numbers and lists of six real numbers. In a paper by Johnson, Laffey, and Loewy, it was shown that the RNIEP and the SNIEP are different problems. We have shown that the smallest list size in which the RNIEP and the SNIEP are different is five.

A matrix $A = [a_{ij}]$ is said to be subordinate to a given graph $G$ if whenever vertex $v_i$ and $v_j$ are not adjacent then entry $a_{ij} = 0$. We resolved some cases of the NIEP where the realizing nonnegative matrix was subordinate to a bipartite graph.

In conclusion, the results in this dissertation have taken us several steps closer to the resolution of one of the outstanding problems in matrix theory.

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# CHAPTER 1

## Introduction

We will be studying the nonnegative inverse eigenvalue problem (NIEP). The NIEP is to determine necessary and sufficient conditions for a list of $n$ complex numbers to be the spectrum of a nonnegative $n$ x $n$ matrix. While this problem is simple and very easy to understand, it remains unsolved. This problem has been the subject of much study over the last 50 years giving various partial solutions. In this dissertation we resolve some of the special cases of the NIEP.

The next seven chapters of this dissertation present a potpourri of results. Because of the diversity of material presented, we dedicate Chapters 2, 3, and 5 to presenting background information regarding the subproblems for which we have new results. We present our new results in Chapters 4, 6, 7, and 8.

We begin with a discussion of nonnegative matrices with various known pertinent results and theorems in Chapter 2. In Chapter 3, we examine the development of the NIEP. In this chapter we give the results of Suleimanova, Perfect, Loewy, and London regarding their solutions to subproblems of the NIEP. Here we see the solutions to the 2 x 2 and 3 x 3 NIEP as well as the trace zero solutions to a list of four or five complex numbers. We then examine the history of the real and symmetric nonnegative inverse eigenvalue problems in Chapter 5. Here we have the results of Johnson et.al. regarding

the difference between the two problems. We present the sufficient conditions of Salzmann, Kellogg, Fiedler, Borobia, and Radwan regarding the real nonnegative inverse eigenvalue problem (RNIEP) or the symmetric nonnegative inverse eigenvalue problem (SNIEP), and conclude with a brief discussion on the Soules construction of a symmetric nonnegative matrix.

We present new results in Chapters 4, 6, 7, and 8. In Chapter 4, we present the partial solution to the NEIP in regard to a list of four complex numbers with nonzero trace with our main result in Theorem 14. Chapter 6 contains the analysis of a list of five real numbers and shows that this is the smallest list in which the two problems, the RNIEP and SNIEP, are different. This is presented in Corollary 4. In Chapter 7, we analyze the list of six real numbers relative to the RNIEP and SNIEP and resolve many cases. Finally, the last chapter of results, Chapter 8, contains a partial solution to the question of when a list of complex numbers that sums to zero is the spectrum of a nonnegative matrix subordinate to a bipartite graph.

# CHAPTER 2

## Nonnegative Matrices

Nonnegative matrices are used in many areas of pure and applied mathematics. Numerical analysis, statistics, molecular chemistry, and economics are examples of areas where nonnegative matrices appear. For instance, the Leontief input-output matrix in economics is a nonnegative matrix. In combinatorics, the adjacency matrix of a graph is nonnegative. A matrix where each entry represents the probability of occurrence is nonnegative.

Throughout this dissertation we will restrict our attention to the matrices that are square with real or complex entries. We will define a matrix $A$ to be a *nonnegative matrix* if every entry of $A$ is real and nonnegative, and a matrix $A$ to be a *positive matrix* if every entry of $A$ is real and positive. We will denote a nonnegative matrix $A$ by $A \geqslant 0$ and a positive matrix $A$ by $A > 0$. An order relation between two real matrices, $A$ and $B$, will be defined as follows. We write $A \geqslant B$ if $A - B \geqslant 0$ and $A > B$ if $A - B > 0$. Furthermore, given $A = [a_{ij}]$, we define $|A|$ to be the matrix containing the absolute values of the entries of $A$. That is, $|A| = [|a_{ij}|]$. Note that for any $A$, $|A| \geqslant 0$.

The notation $M_n$ will denote the set of all $n$ x $n$ matrices with complex entries. Given a matrix $A \in M_n$ and $x \in C^n$, we consider solutions to the equation $Ax = \lambda x$,

where $x \neq 0$ and $\lambda$ is a complex number. If a number $\lambda$ and a nonzero vector $x$ satisfy this equation, then $\lambda$ is called an *eigenvalue* of $A$ and $x$ is called an *eigenvector* of $A$ associated with $\lambda$. Notice that the collection of eigenvectors associated with $\lambda$ and the zero vector form a subspace of $C^n$ called the *eigenspace of* $\lambda$. Associated with the matrix $A$ is an $n^{th}$ degree polynomial $p(\lambda) = \det(\lambda I - A)$. The eigenvalues of $A$ may also be defined as the $n$ roots of the polynomial $p(\lambda)$. We call this polynomial the *characteristic polynomial* of $A$. The set of eigenvalues of $A$ is called the *spectrum* of $A$ and is denoted $\sigma(A)$. The spectral radius of $A$, denoted $\rho(A)$, is the nonnegative real number defined by $\rho(A) = max\{|\lambda| : \lambda \in \sigma(A)\}$.

In this dissertation we will use the following vector norms on $C^n$.

1. the *sum norm* or $l_1$ *norm* on $C^n$ given by $\|x\|_1 \equiv |x_1| + \cdots + |x_n|$,

2. the *max norm* or $l_\infty$ *norm* on $C^n$ given by $\|x\|_\infty \equiv max\{|x_1|, \ldots, |x_n|\}$, and

3. the *Euclidean norm* or $l_2$ *norm* on $C^n$ given by $\|x\|_2 \equiv \left(\sum_{i=1}^n |x_i|^2\right)^{\frac{1}{2}}$.

The unit ball $\{x : \|x\| \leqslant 1\}$ of $C^n$ is compact in any of the vector norms.

Associated with each vector norm on $C^n$ is a natural matrix norm on $M_n$ that is induced by the vector norm as follows. Define $\|\cdot\|$ on $M_n$ by $\|A\| \equiv max_{\|x\|=1}\|Ax\|$. One such norm is the *spectral norm* or the *operator norm* induced by the Euclidean norm on $C^n$. This spectral norm is denoted $\|A\|_2$ and is equivalently defined as $\|A\|_2 = max\{\sqrt{\lambda} : \lambda \in \sigma(A^*A)\}$. It is known that for any matrix norm $\|A\| \geqslant \rho(A)$ and the spectral radius formula is given by $\rho(A) = \lim\limits_{k\to\infty} \|A^k\|^{\frac{1}{k}}$.

When investigating the spectrum of a given matrix, it is often useful to find special properties of the spectrum without finding all the eigenvalues. Note that for the following real matrix

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

the spectrum $\sigma(A) = \{i, -i\}$ is not real. If we consider another example

$$B = \begin{bmatrix} -1 & -1 \\ 2 & -4 \end{bmatrix},$$

then the spectrum $\sigma(B) = \{-2, -3\}$ is real and negative. In neither case does the spectral radius $\rho(A) = 1$ and $\rho(B) = 3$ belong to the respective spectrum.

Considering the fact that real matrices may have only complex eigenvalues or only nonpositive eigenvalues, it is remarkable to note that a nonnegative matrix must have at least one positive eigenvalue. More precisely, the spectral radius of a nonnegative matrix must be an eigenvalue.

We begin the discussion of nonnegative matrices with the result of O. Perron [26]. In 1907, Perron studied positive square matrices. He found that a positive square matrix always has its spectral radius as an element of the spectrum. Therefore we call the spectral radius of a positive or nonnegative matrix $A$ the *Perron root.* An eigenvector associated with the Perron root is called the *Perron vector*. The following due to Perron [26] states the conditions on both the Perron root and the Perron vector.

**Theorem 1 (Perron's Theorem).** [11] *If $A \in M_n$ and $A > 0$, then*

1. $\rho(A) > 0$;

2. $\rho(A)$ *is an eigenvalue of* $A$;

3. *there is an* $x \in C^n$ *with* $x > 0$ *and* $Ax = \rho(A)x$;

4. $\rho(A)$ *is an algebraically (and hence geometrically) simple eigenvalue of* $A$;

5. $|\lambda| < \rho(A)$ *for every eigenvalue* $\lambda \neq \rho(A)$, *that is,* $\rho(A)$ *is the unique eigenvalue of maximum modulus.*

Notice that in addition to the Perron root being an element of the spectrum, both the Perron root and the Perron vector of a positive matrix must be positive.

G. Frobenius extended many of Perron's results to irreducible, nonnegative matrices. Recall that a matrix, $A$, is said to be *reducible* if there exists a permutation matrix $P \in M_n$ such that

$$P^T AP = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix}$$

where $B$ and $D$ are square matrices and 0 is a zero matrix. An *irreducible matrix* is therefore a matrix which is not reducible. Frobenius [9] proved that for an irreducible, nonnegative matrix, $A$, $\rho(A)$ is positive and belongs to the spectrum of $A$. Frobenius also proved that the Perron vector remains positive. All of these results are given in the following theorem called the Perron-Frobenius Theorem.

**Theorem 2 (Perron-Frobenius Theorem).** [11] *Let $A \in M_n$ and suppose that $A$ is irreducible and nonnegative. Then*

1. $\rho(A) > 0$;
2. $\rho(A)$ *is an eigenvalue of* $A$;
3. *there is an* $x \in C^n$ *with* $x > 0$ *such that* $Ax = \rho(A)\, x$; *and*
4. $\rho(A)$ *is an algebraically (and hence geometrically) simple eigenvalue of* $A$.

The result (5) of Perron's Theorem is no longer true when $A \geqslant 0$. When $A \geqslant 0$ and irreducible we have the following result.

**Theorem 3.** [11] *Let $A \in M_n$. Suppose $A$ is nonnegative and irreducible and suppose the set $S = \{\lambda_1 = \rho(A), \lambda_2, \ldots, \lambda_k\}$ of eigenvalues of maximum modulus $\rho(A)$ has exactly $k$ distinct elements. Then each eigenvalue $\lambda_i \in S$ has algebraic multiplicity one, and $S = \{e^{\frac{2\pi i t}{k}} \rho(A) : t = 0, 1, \ldots, k-1\}$. Moreover if $\lambda$ is any eigenvalue of $A$, then $e^{\frac{2\pi i t}{k}} \lambda$ is an eigenvalue for all $t = 0, 1, \ldots, k-1$.*

Notice that the necessary requirements, (1) the Perron root being positive and belonging to the spectrum and (2) the Perron vector being positive, are maintained as in Perron's Theorem but now the matrix no longer must be strictly positive. This raises a natural question. Can we remove the irreducibility condition from the Perron-Frobenius Theorem? In order to investigate this question, we will need Theorem 4.

**Theorem 4.** [11] *Let $A,\ B \in M_n$. If $|A| \leqslant B$, then*

$\rho(A) \leqslant \rho(|A|) \leqslant \rho(B)$. *In particular if $0 \leqslant A \leqslant B$, then $\rho(A) \leqslant \rho(B)$.*

*Proof.* For every $m = 1, 2, \ldots$ we have $|A^m| \leqslant |A|^m \leqslant B^m$. Hence for the spectral norm, $\|A^m\|_2 \leqslant \| \, |A|^m \, \|_2 \leqslant \|B^m\|_2$ for all $m = 1, 2, \ldots$. By using the spectral radius formula we deduce that $\rho(A) \leqslant \rho(|A|) \leqslant \rho(B)$. If $0 \leqslant A \leqslant B$, then $\rho(A) \leqslant \rho(B)$. □

In the Perron-Frobenius Theorem, the requirement of the matrix $A$ being irreducible as well as nonnegative led to the necessary condition that the Perron vector is positive. When the matrix $A$ is just nonnegative, the condition becomes that the Perron vector is nonnegative. The condition that $\rho(A)$ be an element of the spectrum still holds true. These facts are proved in Theorem 5.

**Theorem 5.** [11] *If $A \in M_n$ and $A \geqslant 0$, then $\rho(A)$ is an eigenvalue of $A$ and there is a nonnegative vector $x \geqslant 0$, $x \neq 0$, such that $Ax = \rho(A)\, x$.*

*Proof.* Let $\varepsilon > 0$. Define $A(\varepsilon) \equiv [a_{ij} + \varepsilon] > 0$. Let $x(\varepsilon)$ represent the Perron vector of $A(\varepsilon)$ such that $x(\varepsilon) > 0$ and $\sum_{i=1}^{n} x(\varepsilon)_i = 1$. As the set of vectors $\{x(\varepsilon) : \varepsilon > 0\}$ is contained in the compact set $\{x : x \in C^n, \|x\|_1 \leqslant 1\}$, there is a monotone decreasing sequence $\varepsilon_1,\ \varepsilon_2, \ldots$ with $\lim_{k\to\infty} \varepsilon_k = 0$ such that $\lim_{k\to\infty} x(\varepsilon_k) \equiv x$ exists. Since $x(\varepsilon_k) > 0$ for all $k = 1,\ 2,\ \ldots$, it must be that $x = \lim_{k\to\infty} x(\varepsilon_k) \geqslant 0$. Note, $x = 0$ is impossible since $\sum_{i=1}^{n} x_i = \lim_{k\to\infty} \sum_{i=1}^{n} x(\varepsilon_k)_i \equiv 1$. By Theorem 4, we have $\rho(A(\varepsilon_k)) \geqslant \rho(A(\varepsilon_{k+1})) \geqslant \cdots \geqslant \rho(A)$ for all $k = 1,\ 2,\ \ldots$. So the sequence of real numbers $\{\rho(A(\varepsilon_k))\}_{k=1,2,\ldots}$ is

a monotone decreasing sequence. Therefore $\rho \equiv \lim_{k\to\infty} \rho\left(A\left(\varepsilon_k\right)\right)$ exists and $\rho \geqslant \rho\left(A\right)$. Now because $Ax = \lim_{k\to\infty} A\left(\varepsilon_k\right) x\left(\varepsilon_k\right) = \lim_{k\to\infty} \rho\left(A\left(\varepsilon_k\right)\right) x\left(\varepsilon_k\right) = \rho x$ and $x \neq 0$, it follows that $\rho$ is an eigenvalue of $A$ so $\rho \leqslant \rho\left(A\right)$. Thus $\rho = \rho\left(A\right)$. □

The following result will be used in Chapters 4 and 8. We state it here for reference.

Define a matrix $B$ as follows

$$B = \begin{bmatrix} 0 & B_{1,2} & 0 & \cdots & 0 \\ 0 & 0 & B_{2,3} & \cdots & 0 \\ & & \ddots & & \\ 0 & 0 & \cdots & 0 & B_{k-1,k} \\ B_{k,1} & 0 & \cdots & 0 & 0 \end{bmatrix}$$

where the block $B_{j,j+1}$ is $n_j$ x $n_{j+1}$ for $j = 1, \ldots, k-1$ and $B_{k,1}$ is $n_k$ x $n_1$ to be in *super diagonal form* [22]. This form has a special relationship with its eigenvalues as Theorem 6 shows.

**Theorem 6.** [22] *Let $A$ be an $n$ x $n$ complex matrix in super diagonal form and suppose that $\omega_1, \ldots, \omega_m$ are the nonzero eigenvalues of the product $A_{1,2}A_{2,3}\cdots A_{k,1}$. The spectrum of $A$ consists of the $n - km$ zeros and the $km$ $k^{th}$ roots of the numbers $\omega_1, \ldots, \omega_m$.*

The discoveries of Perron and Frobenius have given some powerful necessary conditions in the study of nonnegative matrices. The spectrum of a nonnegative matrix must contain the Perron root which is positive in all cases except the zero matrix where it is zero, and the Perron vector must be nonnegative. For example we can

deduce that a list of negative numbers will never be the spectrum of any nonnegative matrix. This is because the Perron root is not part of the set. So addressing the special properties of nonnegative matrices has become a source of interesting mathematical problems, we will explore some of these ideas in the remaining chapters.

CHAPTER 3

Nonnegative Inverse Eigenvalue Problem

## 3.1 Beginning of the Nonnegative Inverse Eigenvalue Problem

In 1938 Kolmogorov asked when is a given complex number an eigenvalue of a nonnegative matrix. When there is no restriction on the order of the matrix, then any complex number is an eigenvalue of some nonnegative matrix. A simple solution comes from Minc [22]: if $\alpha + \beta i$, where $\alpha$ and $\beta$ are any real numbers, is the complex number in question, then the *circulant* matrix

$$A(\alpha, \beta) = \begin{bmatrix} |\alpha| & |\beta| & 0 & 0 \\ 0 & |\alpha| & |\beta| & 0 \\ 0 & 0 & |\alpha| & |\beta| \\ |\beta| & 0 & 0 & |\alpha| \end{bmatrix}$$

will have $|\alpha| \pm |\beta|$ and $|\alpha| \pm |\beta| i$ as eigenvalues. Now let $B(\alpha, \beta)$ be the 8 x 8 matrix

$$B(\alpha, \beta) = \begin{bmatrix} 0 & A(\alpha, \beta) \\ A(\alpha, \beta) & 0 \end{bmatrix}.$$

Theorem 6 shows that the eigenvalues of $B(\alpha, \beta)$ are $\pm |\alpha| \pm |\beta|$ and $\pm |\alpha| \pm |\beta| i$. Note that one of the last four eigenvalues is the given $\alpha + \beta i$.

It can be easily shown that a single complex number with imaginary part not equal to zero cannot be an eigenvalue of nonnegative matrices of order 1 or order 2.

In the case of an order 1 nonnegative matrix, there is exactly one eigenvalue, namely the Perron root. Since any complex root of a characteristic polynomial with real coefficients must occur with its conjugate pair, in the case of an order 2 nonnegative matrix we know that if one root is complex then so is the second. But one root must be the real Perron root and so the second must be real also, thereby eliminating the possibility of a complex root. Therefore, the smallest dimension for which a complex number with nonzero imaginary part can be an element of the spectrum of a nonnegative matrix is $n = 3$. The general problem as set above by Kolmogorov without any restriction on the matrix thus has a complete solution. To extend this problem it becomes necessary to restrict the question to certain classes of nonnegative matrices: e.g. stochastic matrices, doubly stochastic matrices, nonnegative circulants. Recall that a *stochastic matrix* is a nonnegative matrix with all row sums equal to one, and a *doubly stochastic matrix* is a stochastic matrix with the added restriction that all column sums equal one. History credits Dmitriev and Dynkin [5, 6] for partial results in the case of stochastic matrices and Karpelevich [14] for completely solving it.

## 3.2 Growth of the Nonnegative Inverse Eigenvalue Problem

In 1949 Suleimanova [32] extended Kolmogorov's question to what is now called the nonnegative inverse eigenvalue problem (NIEP). *The NIEP is to determine necessary and sufficient conditions for a list of* $n$ *complex numbers to be the spectrum of a nonnegative* $n \times n$ *matrix.* Suleimanova answered this question when the list of length

$n$ consists of only real numbers where exactly one number is positive. Suleimanova's sufficient condition for $\sigma = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ where $\lambda_1 > 0 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$ to be the spectrum of a nonnegative matrix is that $\sum_{j=1}^{n} \lambda_j \geqslant 0$. We will prove Suleimanova's result in Chapter 5. But note for now that this result can be extended to any list of real numbers $\sigma = \{\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n\}$ such that $\lambda_1 + \sum_{\lambda_j < 0} \lambda_j \geqslant 0$. Here the Perron root dominates the sum of all the negative numbers. This extension is due to Perfect and is discussed further in Chapter 5.

## 3.3 Necessary Conditions for the NIEP

If a matrix $A$ is nonnegative then its trace must be nonnegative. Since powers of $A$ are nonnegative so are their traces. This necessary condition is described as follows: Let $A$ be an $n$ x $n$ nonnegative matrix. Suppose $\sigma = \{\lambda_1, \ldots, \lambda_n\}$ is the list of eigenvalues (repeats included) of $A$. Then define the $k^{th}$ moment $s_k = \sum_{i=1}^{n} \lambda_i^k = trace\left(A^k\right)$ for $k = 1,\ 2,\ \ldots$. The necessary condition can be stated as $s_k \geqslant 0$ for $k = 1, 2, \ldots$.

Since the characteristic polynomial of a nonnegative matrix has only real coefficients, the complex eigenvalues must occur in conjugate pairs. This is indicated by the necessary condition $\bar{\sigma} = \sigma$ where $\bar{\sigma} = \{\bar{\lambda_1}, \ldots, \bar{\lambda_n}\}$.

The third condition due to Johnson, Loewy and London is described in the following theorem. The condition appeared first in Loewy and London [20]. Johnson independently found the condition in [12]. We will henceforth call this necessary condition the JLL condition.

**Theorem 7.** [20] *Let $\sigma = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ be the set of eigenvalues of a nonnegative $n$ x $n$ matrix $A = [a_{ij}]$. Then $s_k^m \leqslant n^{m-1} s_{km}$ for $k, m = 1, 2, \ldots$.*

*Proof.* Consider the diagonal matrix $D = diag\,(a_{11}, a_{22}, a_{33}, \ldots, a_{nn})$ and let $A = C + D$. Since $A$ is nonnegative so are $C$ and $D$. It follows that $A^m - C^m - D^m$ for $m = 1, 2, \ldots$ is nonnegative too. This gives $trace\,(A^m) \geqslant trace\,(C^m) + trace\,(D^m) = trace\,(C^m) + \sum_{i=1}^{n} a_{ii}^m$.

Then by Holder's inequality we have, $\left(\sum_{i=1}^{n} a_{ii}\right)^m \leqslant n^{m-1} \sum_{i=1}^{n} a_{ii}^m$. Therefore,

$$\begin{aligned}
trace\,(A^m) &= s_m \\
&\geqslant trace\,(C^m) + \sum_{i=1}^{n} a_{ii}^m \\
n^{m-1} s_m &\geqslant n^{m-1} trace\,(C^m) + n^{m-1} \sum_{i=1}^{n} a_{ii}^m \\
&\geqslant n^{m-1} trace\,(C^m) + \left(\sum_{i=1}^{n} a_{ii}\right)^m \\
&= n^{m-1} trace\,(C^m) + (s_1)^m
\end{aligned}$$

Which results in $n^{m-1} s_m - s_1^m \geqslant n^{m-1} trace\,(C^m)$. But since $trace\,(C^m) \geqslant 0$ this implies $n^{m-1} s_m - s_1^m \geqslant 0$ which then gives $n^{m-1} s_m \geqslant s_1^m$ for $m = 1, 2, \ldots$. Applying this inequality (which is true for $k = 1$) to the nonnegative matrix $A^k$, we get $s_k^m \leqslant n^{m-1} s_{km}$ for all $k, m = 1, 2, \ldots$.

$\square$

Given a list $\sigma = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, we summarize the four necessary conditions for the NIEP below.

1. If $\rho = max_{1 \leqslant i \leqslant n}\{|\lambda_i| : \lambda_i \in \sigma\}$, then $\rho \in \sigma$ (Perron root condition),

2. $\bar{\sigma} = \sigma$, i.e., $\sigma$ must be closed under conjugation,

3. $\sum_{i=1}^{n} \lambda_i^k \geqslant 0$ for $k = 1, 2, \ldots$, i.e., all moments are nonnegative, and

4. the JLL condition: $s_k^m \leqslant n^{m-1} s_{km}$ for all $k, m = 1, 2, \ldots$.

We say that a list $\sigma = \{\lambda_1, \ldots, \lambda_n\}$ of length $n$ is *realizable* by an $n$ x $n$ nonnegative (resp. symmetric nonnegative) matrix if there exists a matrix $A \geqslant 0$ (resp. symmetric matrix $A \geqslant 0$) such that $\sigma(A)$ is the given list $\sigma$.

It is easy to give examples that conditions (1), (2), and (3) are not sufficient. Let $\sigma = \{1, 1, -\frac{2}{3}, -\frac{2}{3}, -\frac{2}{3}\}$. By the Perron-Frobenius Theorem, since the maximal element is not simple, the realizing matrix must be reducible. However, the given list $\sigma$ cannot be divided into two sublists each satisfying all the necessary conditions (1), (2), and (3).

Now condition (4) does not follow from conditions (1), (2), and (3). We see this by an example. Suppose $\sigma = \{\sqrt{2}, i, -i\}$, then (1) - (3) are satisfied but (4) fails for $k = 1$ and $m = 2$.

For $n \geqslant 4$ condition (4) is not sufficient. To see this consider $\sigma = \{\sqrt{2}, \sqrt{2}, i, -i\}$. It is easy to check that $\sigma$ satisfies conditions (1) - (4). If $\sigma$ is the set of eigenvalues of a nonnegative matrix $A$, then according to the Perron-Frobenius Theorem $A$ is

reducible. Hence $\{\sqrt{2}, i, -i\}$ must be realizable by some nonnegative matrix. But we have already seen that is not possible.

## 3.4 Stochastic Inverse Eigenvalue Problem

Many subproblems have emerged from the NIEP because of its complexity. The stochastic inverse eigenvalue problem is one of these which is to determine necessary and sufficient conditions for a list of $n$ complex numbers to be the spectrum of a stochastic $n$ x $n$ matrix. The next result shows that when there exists a nonnegative matrix $A$ with spectrum $\sigma = \{\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n\}$ and positive Perron vector, then $\lambda_1^{-1}A$ is diagonally similar to a stochastic matrix.

**Theorem 8.** [22] *If $A$ is a nonnegative matrix with Perron root $\lambda_1$ and a positive Perron vector, then $\lambda_1^{-1}A$ is diagonally similar to a stochastic matrix.*

*Proof.* Let $x = (x_1, x_2, \ldots, x_n)^T$ be the positive Perron vector of $A$. If $D = diag\,(x_1, x_2, \ldots, x_n)$, then $x = Du$ where $u = (1, 1, \ldots, 1)^T$. As 1 is the Perron root of $D^{-1}\left(\lambda_1^{-1}A\right)D = \lambda_1^{-1}D^{-1}AD$, we have $\lambda_1^{-1}D^{-1}ADu = \lambda_1^{-1}D^{-1}Ax = \lambda_1^{-1}D^{-1}\lambda_1 x = D^{-1}x = u$. Therefore $u$ is the Perron vector of
$\lambda_1^{-1}D^{-1}AD = D^{-1}\left(\lambda_1^{-1}A\right)D$. And since a square nonnegative matrix $A$ is stochastic if and only if $u = (1, 1, \ldots, 1)^T$ is an eigenvector corresponding to the Perron root 1 of $A$, we have the desired result. $\square$

Johnson discusses the stochastic inverse eigenvalue problem in [12]. He calls it the *row stochastic eigenvalue problem.* The Perron root of a row stochastic matrix is one;

and, if the Perron root is normalized to be one in the NIEP, then the row stochastic eigenvalue problem is equivalent to the NIEP. (The list $\{0, 0, \ldots, 0\}$ is omitted as we have the zero matrix as a solution.) If $A$ is an arbitrary $n$ x $n$ nonnegative matrix, then $A$ is permutationally similar to

$$P^T AP = \begin{bmatrix} A_1 & & 0 \\ & \ddots & \\ * & & A_m \end{bmatrix}$$

where $A_i$ is $n_i$ x $n_i$, $i = 1, \ldots, m$, and irreducible. The $A_i$ are called the *irreducible components* of $A$ and, of course, if $A$ itself is irreducible, then $m = 1$. Now $\sigma(A) = \bigcup_{i=1}^{m} \sigma(A_i)$ which is the same as $\sigma(A_1 \oplus \cdots \oplus A_m)$. Since each $A_i$ is irreducible, it has a positive Perron vector $x_i$ corresponding to its Perron root $\lambda_i$. If $D_i$ is the positive diagonal matrix obtained by placing the entries of $x_i$ in order, down the diagonal, then $D_i^{-1} A_i D_i$ has all its row sums equal to $\lambda_i$ using Theorem 8. We may assume that $\lambda_i > 0$. If $A$ is initially normalized so that $\lambda_1 = 1 = max_{1 \leqslant i \leqslant m} \lambda_i$, and $A_{i1}$ is arbitrarily chosen to be $n_i$ x $n_1$, $i = 2, \ldots, m$, with nonnegative entries and row sums $1 - \lambda_i$, then

$$\hat{A} = \begin{bmatrix} D_1^{-1} A_1 D_1 & 0 & \cdots & 0 \\ A_{21} & D_2^{-1} A_2 D_2 & \cdots & 0 \\ \vdots & 0 & \ddots & \vdots \\ & \vdots & & \\ A_{m1} & 0 & \cdots & D_m^{-1} A_m D_m \end{bmatrix}$$

is row stochastic and satisfies $\sigma\left(\hat{A}\right) = \sigma(A)$. Thus we have the equivalence of the

row stochastic inverse eigenvalue problem and the (normalized) NIEP.

While the row stochastic inverse eigenvalue problem is equivalent to the NIEP, the doubly stochastic inverse eigenvalue problem, is not. This question is addressed by Johnson. Johnson [12] gave necessary and sufficient conditions on a list of $n$ complex numbers so that it is the spectrum of a doubly stochastic matrix. Recall a *doubly stochastic matrix* is a square matrix that has row sum one and column sum one. The following two observations show that the doubly stochastic inverse eigenvalue problem is equivalent to the row stochastic inverse eigenvalue problem for the 2 x 2 case, and not equivalent for $n \geqslant 3$.

1. Every 2 x 2 row stochastic matrix is similar to a doubly stochastic matrix.

   *Proof.* A 2 x 2 row stochastic matrix has the general form

$$\begin{bmatrix} \alpha & 1-\alpha \\ \beta & 1-\beta \end{bmatrix}$$

   for $0 \leqslant \alpha, \beta \leqslant 1$, which is similar to

$$\begin{bmatrix} \frac{1-\beta+\alpha}{2} & \frac{1-\alpha+\beta}{2} \\ \frac{1-\alpha+\beta}{2} & \frac{1-\beta+\alpha}{2} \end{bmatrix},$$

   a doubly stochastic matrix. □

2. There exist row stochastic matrices that are not similar to any doubly stochastic matrix for order 3 or more.

*Proof.* Consider the following row stochastic matrix,

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

$A$ is not similar to any doubly stochastic matrix. For if one did exist, it would have the form

$$B = \begin{bmatrix} a & b & 1-a-b \\ c & d & 1-c-d \\ 1-a-c & 1-b-d & a+b+c+d-1 \end{bmatrix}.$$

So for $A$ and $B$ to be similar, both the traces and the 2 x 2 principal minor sums would have to be equal. Thus,

(a) $2\,(a+d)+(b+c)-1=0$ and

(b) $3\,(ad-bc)+(a+d)+2\,(b+c)-2=-1.$

But since the (3,3) entry of $B$ must be nonnegative, this means $a+d \geqslant 1-(b+c)$. By (a), we have $2\,(a+d)=1-(b+c)$ resulting in the conclusion that $a+d=0$ implying $a=0$ and $d=0$. This means that $b+c=1$. This makes (b) become $3\,(-bc)=-3b(1-b)=-1$ or equivalently $b(1-b)=\frac{1}{3}$. This has no solution for $0 \leqslant b \leqslant 1$ showing that such a $B$ cannot exist. □

For more information regarding the doubly stochastic matrix see [12]. We will now return to the NIEP and discuss some known solutions.

## 3.5 2 x 2 and 3 x 3 NIEP Solutions

Two cases of the NIEP are completely solved. The two cases are when $\sigma = \{\lambda_1, \lambda_2\}$ and when $\sigma = \{\lambda_1, \lambda_2, \lambda_3\}$. We will begin when the set $\sigma$ contains two numbers. We observed in this case $\lambda_1$ and $\lambda_2$ must be real.

The list $\sigma = \{\lambda_1,\ \lambda_2\}$ must satisfy $\lambda_1 + \lambda_2 \geqslant 0$ where $\lambda_1 \geqslant |\lambda_2|$. The following (symmetric) nonnegative matrix realizes $\sigma$:

$$A = \frac{1}{2}\begin{bmatrix} \lambda_1 + \lambda_2 & \lambda_1 - \lambda_2 \\ \lambda_1 - \lambda_2 & \lambda_1 + \lambda_2 \end{bmatrix}.$$

Now in the case when $\sigma = \{\lambda_1, \lambda_2, \lambda_3\}$, Loewy and London [20] found the necessary and sufficient conditions. They are given in Theorem 9.

**Theorem 9.** [20] *Let $\sigma = \{\lambda_1, \lambda_2, \lambda_3\}$ be a list of three complex numbers, and assume that $\sigma$ satisfies the following conditions*

1. $\max\limits_{1 \leqslant i \leqslant 3}\{|\lambda_i| : \lambda_i \in \sigma\} \in \sigma$,

2. $\bar{\sigma} = \sigma$,

3. $s_1 = \lambda_1 + \lambda_2 + \lambda_3 \geqslant 0$, *and*

4. $s_1^2 \leqslant 3s_2$.

*Then $\sigma$ is realized by a nonnegative matrix $A$.*

*Proof.* If $\sigma$ is real, the result will follow from Suleimanova's result or Perfect's result mentioned earlier.

If $\sigma$ is complex, then $\sigma = \{\rho e^{i\theta}, \rho e^{-i\theta}, r\}$, where $0 < \theta < \pi$ and $0 < \rho \leqslant r$. As we know the hypotheses (1)-(4) are homogeneous, we may assume $\rho = 1$. This gives $\sigma = \{e^{i\theta}, e^{-i\theta}, r\}$, where $0 < \theta < \pi$ and $r \geqslant 1$.

Now, the trace condition $s_1 = \lambda_1 + \lambda_2 + \lambda_3 \geqslant 0$ and $\sigma = \{e^{i\theta}, e^{-i\theta}, r\}$ imply $2\cos\theta + r \geqslant 0$; and substituting into $s_1^2 \leqslant 3s_2$ the information $\sigma = \{e^{i\theta}, e^{-i\theta}, r\}$, implies $3\left(2\cos 2\theta + r^2\right) - \left(2\cos\theta + r\right)^2 \geqslant 0$.
Thus, $\left(r - 2\cos\left(\frac{1}{3}\pi + \theta\right)\right)\left(r - 2\cos\left(\frac{1}{3}\pi - \theta\right)\right) \geqslant 0$ follows.

But for $0 < \theta < \pi$, $r \geqslant 1$, we have $r - 2\cos\left(\frac{1}{3}\pi + \theta\right) \geqslant 0$ which means $r - 2\cos\left(\frac{1}{3}\pi - \theta\right) \geqslant 0$.

So, if $\sigma$ is of the form $\sigma = \{e^{i\theta}, e^{-i\theta}, r\}$, then our last two hypotheses become

1. $2\cos\theta + r \geqslant 0$ and

2. $r - 2\cos\left(\frac{1}{3}\pi - \theta\right) \geqslant 0$ respectively.

Now construct a nonnegative matrix $A$ having $\sigma = \{e^{i\theta}, e^{-i\theta}, r\}$ as its spectrum. Let $U$ be the orthogonal matrix

$$U = \frac{1}{\sqrt{6}}\begin{bmatrix} \sqrt{2} & \sqrt{3} & -1 \\ \sqrt{2} & 0 & 2 \\ \sqrt{2} & -\sqrt{3} & -1 \end{bmatrix}$$

and let

$$J = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Then we have,

$$U^T JU = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

So $\sigma = \{e^{i\theta}, e^{-i\theta}, r\}$ is the spectrum of the real matrix

$$A' = \begin{bmatrix} r & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}.$$

Therefore, by the matrix multiplication

$$\begin{aligned} A = UA'U^T &= \\ &= U \begin{bmatrix} r & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} U^T + U \begin{bmatrix} 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} U^T \\ &= \frac{1}{3} rJ \\ &- \frac{2}{3} \begin{bmatrix} -\cos\theta & \cos\left(\frac{1}{3}\pi + \theta\right) & \cos\left(\frac{1}{3}\pi - \theta\right) \\ \cos\left(\frac{1}{3}\pi - \theta\right) & -\cos\theta & \cos\left(\frac{1}{3}\pi + \theta\right) \\ \cos\left(\frac{1}{3}\pi + \theta\right) & \cos\left(\frac{1}{3}\pi - \theta\right) & -\cos\theta \end{bmatrix}. \end{aligned}$$

It follows that $A$ is nonnegative. $\square$

The following geometrical interpretation of Theorem 9 comes from the work of Dmitriev and Dynkin [5, 6].

**Corollary 1.** [5, 6] *$\sigma = \{r, \rho e^{i\theta}, \rho e^{-i\theta}\}$ where $0 \leqslant \rho \leqslant r$ is the set of eigenvalues of a nonnegative matrix if and only if $\rho e^{i\theta}$ belongs to the closed triangle whose vertices are $r, re^{\frac{2\pi i}{3}}, re^{-\frac{2\pi i}{3}}$.*

This ends the solution of the NIEP for any lists of length two or three. The next section provides the solution to lists of length four or five when the sum of the numbers in the list is zero. This is called the trace zero NIEP.

### 3.6 Trace Zero NIEP

A special set of matrices are those with trace zero. If $A \geqslant 0$ and $trace\,(A) = 0$, then it follows that the main diagonal entries must be zero. Reams [29] completely solved the trace zero NIEP for lists of length four as we see in the following Theorem 10.

**Theorem 10.** [29] *Let $\sigma = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ be a list of four complex numbers. If $s_1 = 0$, $s_2 \geqslant 0$, $s_3 \geqslant 0$, and $4s_4 \geqslant s_2^2$, then there exists a nonnegative* 4 x 4 *matrix with spectrum $\sigma$.*

*Proof.* Notice that the last inequality among the hypotheses of the theorem is the JLL condition with $n = 4$, $k = 2$, and $m = 2$.

Let

$$p_1 = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4,$$

$$p_2 = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4,$$

$$p_3 = \lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_4 + \lambda_1\lambda_3\lambda_4 + \lambda_2\lambda_3\lambda_4, \text{ and}$$

$$p_4 = \lambda_1\lambda_2\lambda_3\lambda_4.$$

i.e. $p_1, p_2, p_3, p_4$ are Newton's elementary symmetric polynomials.

Newton's identities for symmetric functions [33] state that

$$s_1 - p_1 = 0,$$

$$s_2 - p_1 s_1 + 2p_2 = 0,$$

$$s_3 - p_1 s_2 + p_2 s_1 - 3p_3 = 0, \text{ and}$$

$$s_4 - p_1 s_3 + p_2 s_2 - p_3 s_1 + 4p_4 = 0.$$

We can write these equations in matrix form with a companion matrix as

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -p_4 & p_3 & -p_2 & p_1 \end{bmatrix} \begin{bmatrix} -4 & 0 & 0 & 0 \\ -s_1 & -3 & 0 & 0 \\ -s_2 & -s_1 & -2 & 0 \\ -s_3 & -s_2 & -s_1 & -1 \end{bmatrix} = \begin{bmatrix} -s_1 & -3 & 0 & 0 \\ -s_2 & -s_1 & -2 & 0 \\ -s_3 & -s_2 & -s_1 & -1 \\ -s_4 & -s_3 & -s_2 & -s_1 \end{bmatrix}$$

Letting $p_1 = s_1 = 0$ and multiplying both sides on the right by

$diag\left(-\frac{1}{4}, -\frac{1}{3}, -\frac{1}{2}, -1\right)$ we get

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -p_4 & p_3 & -p_2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{s_2}{4} & 0 & 1 & 0 \\ \frac{s_3}{4} & \frac{s_2}{3} & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{s_2}{4} & 0 & 1 & 0 \\ \frac{s_3}{4} & \frac{s_2}{3} & 0 & 1 \\ \frac{s_4}{4} & \frac{s_3}{3} & \frac{s_2}{2} & 0 \end{bmatrix}. \tag{1}$$

It is easily verified that

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{s_2}{4} & 0 & 1 & 0 \\ \frac{s_3}{4} & \frac{s_2}{3} & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{s_2}{4} & 0 & 1 & 0 \\ -\frac{s_3}{4} & -\frac{s_2}{3} & 0 & 1 \end{bmatrix}$$

then multiplying on the left of (1) with this matrix we get

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{s_2}{4} & 0 & 1 & 0 \\ -\frac{s_3}{4} & -\frac{s_2}{3} & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -p_4 & p_3 & -p_2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{s_2}{4} & 0 & 1 & 0 \\ \frac{s_3}{4} & \frac{s_2}{3} & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{s_2}{4} & 0 & 1 & 0 \\ \frac{s_3}{4} & \frac{s_2}{12} & 0 & 1 \\ \frac{3s_4-s_2^2}{12} & \frac{s_3}{12} & \frac{s_2}{6} & 0 \end{bmatrix}.$$

Finally, performing a similarity on this matrix we have

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \frac{s_2}{12} & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{s_2}{4} & 0 & 1 & 0 \\ \frac{s_3}{4} & \frac{s_2}{12} & 0 & 1 \\ \frac{3s_4-s_2^2}{12} & \frac{s_3}{12} & \frac{s_2}{6} & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -\frac{s_2}{12} & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{s_2}{4} & 0 & 1 & 0 \\ \frac{s_3}{4} & 0 & 0 & 1 \\ \frac{4s_4-s_2^2}{16} & \frac{s_3}{12} & \frac{s_2}{4} & 0 \end{bmatrix},$$

which is nonnegative and similar to the companion matrix with eigenvalues $\lambda_1, \lambda_2, \lambda_3,$ $\lambda_4$ as required. □

Reams gave sufficient conditions for the trace zero NIEP for the lists of five numbers. The 5 x 5 trace zero NIEP was completely solved by Laffey and Meehan [18]. Since the proof involves results from combinatorics, we merely state the results.

**Theorem 11.** [18] *Let* $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$ *be complex numbers and let* $s_k = {\lambda_1}^k + {\lambda_2}^k + {\lambda_3}^k + {\lambda_4}^k + {\lambda_5}^k$ *for* $k = 1, 2, \ldots$ . *Assume* $s_1 = 0$. *Then* $\sigma = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\}$ *is the spectrum of a nonnegative* 5 x 5 *matrix if and only if the following conditions hold:*

1. $s_k \geqslant 0$ *for* $k = 1, 2, 3, 4, 5$,

2. $4s_4 \geqslant s_2^2$, *and*

3. $12s_5 - 5s_2s_3 + 5s_3\sqrt{4s_4 - s_2^2} \geqslant 0$.

A refinement of the JLL condition in the case of trace zero lists of odd length comes from Laffey and Meehan [17]. Laffey and Meehan's result above uses this refinement in condition (2).

**Theorem 12.** [17] *Let* $A$ *be a nonnegative* $n$ x $n$ *matrix with* $trace\,(A) = 0$. *Then, if* $n$ *is odd,* $(n-1)\, s_4 \geqslant s_2^2$, *that is,* $(n-1)\, trace\,(A^4) \geqslant (trace\,(A^2))^2$.

This inequality is the best possible. Consider the matrix for odd $n$,

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus [0],$$

then equality occurs. For $n$ even, the matrix

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

gives an example that shows Theorem 12 fails when $n$ is even.

This concludes the known results regarding the NIEP for lists of length $n \leqslant 5$. In the next chapter will return to the list of four complex numbers in an attempt to solve the general NIEP.

# CHAPTER 4

## Lists of Length Four

In this chapter, we will examine a list of four complex numbers. We first review what has been solved in the smaller dimensions.

The 1 x 1 case is trivial. Clearly the eigenvalue must be nonnegative to satisfy the Perron root condition and so 1 x 1 nonnegative matrix is the Perron root. Now, when the list contains two elements, the eigenvalues must be real. This is because the characteristic polynomial of a nonnegative matrix has real coefficients, so the eigenvalues must occur in conjugate pairs. As one must be the Perron root, the other cannot be a complex number with nonzero imaginary part. We have seen the solution to this case in Chapter 3.

Now in the case where the list has three elements, we have seen in Chapter 3 that Theorem 9 by Loewy and London solves the NIEP. The conditions from Theorem 9 are not sufficient for the list of four complex numbers. To see this, consider $\sigma = \{\sqrt{2}, \sqrt{2}, i, -i\}$. As this set satisfies all the necessary conditions, it appears to be a good candidate for the spectrum of a 4 x 4 nonnegative matrix. But according to the Perron-Frobenius Theorem, if $\sigma$ is realizable by a nonnegative matrix then it is reducible. The only possible partition requires the grouping $\tilde{\sigma} = \{i, -i, \sqrt{2}\}$. But this $\tilde{\sigma}$ fails the necessary requirements to be a spectrum of a 3 x 3 nonnegative matrix

(as seen in Chapter 3), so $\sigma$ cannot be the spectrum of a 4 x 4 nonnegative matrix.

Thus in search of sufficient conditions for a solution to the NIEP for lists of four numbers, we divide the problem into two parts. One is when $\sigma$ is real, and the other is when $\sigma$ has two complex numbers which are complex conjugates of each other. The first one has a complete solution given by Loewy and London which will be presented in Chapter 6. But for the second case, when $\sigma = \{\lambda_1, \lambda_2, a + bi, a - bi\}$ where $\lambda_1$ is the Perron root, $\lambda_2$ is real, and $b \neq 0$, no solution is found in the literature. We will begin with the case where we have normalized the spectrum by dividing by $\lambda_1$ and setting $a = 0$.

## 4.1 NIEP for $\sigma = \{1, \lambda, bi, -bi\}$

We will analyze the list $\sigma = \{1, \lambda, bi, -bi\}$. If $b \neq 0$, we know that nonnegative matrices with $\sigma$ as spectrum cannot be symmetric since symmetric matrices have real spectra.

Analysis on the $\lambda$ and $b$ helps us limit the region under consideration. Clearly, as 1 is the Perron root, then both $-1 \leqslant \lambda \leqslant 1$ and $-1 \leqslant b \leqslant 1$ must be true. Moreover, as we have both $bi$ and $-bi$, we may assume $0 < b \leqslant 1$. So when we graph the region, we have a rectangular region in terms of $\lambda$ and $b$ where $-1 \leqslant \lambda \leqslant 1$ and $0 < b \leqslant 1$. Since $\sigma$ must satisfy the the JLL condition, the region for possible $\lambda$ and $b$ becomes smaller. The necessary condition $4s_2 \geqslant s_1^2$ gives $4\left(1 + \lambda^2 - 2b^2\right) \geqslant (1 + \lambda)^2$. This in turn restricts the region to $0 < b \leqslant \sqrt{\frac{3\lambda^2 - 2\lambda + 3}{8}}$ and $-1 \leqslant \lambda \leqslant 1$. No other JLL inequality restricts the region further.

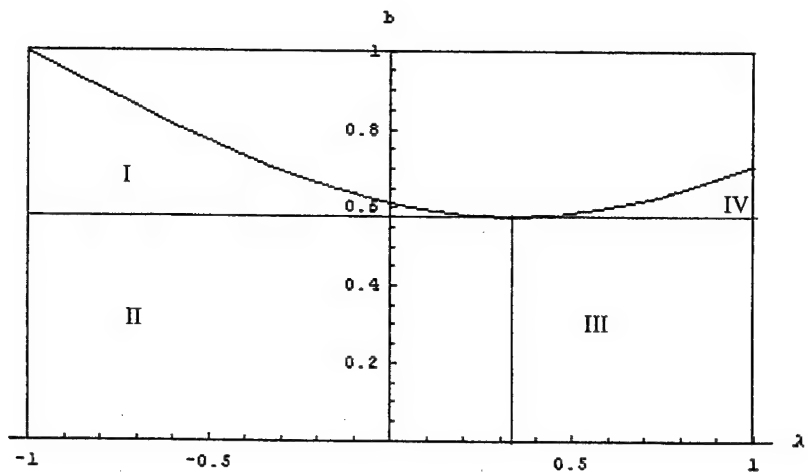


Figure 1: Regions for $\{1, \lambda, bi, -bi\}$

We will denote the regions possible by the following:

1. Region I will be for the area defined when $-1 \leqslant \lambda \leqslant 0$ and $\frac{\sqrt{3}}{3} \leqslant b \leqslant \sqrt{\frac{1}{8}(3\lambda^2 - 2\lambda + 3)}$,

2. Region II will cover the area for $-1 \leqslant \lambda \leqslant 0$ and $0 < b \leqslant \frac{\sqrt{3}}{3}$,

3. Region III has area $0 \leqslant \lambda \leqslant 1$ and $0 < b \leqslant \frac{\sqrt{3}}{3}$, and

4. Region IV covers the area defined by $0 \leqslant \lambda \leqslant 1$ and $\frac{\sqrt{3}}{3} \leqslant b \leqslant \sqrt{\frac{1}{8}(3\lambda^2 - 2\lambda + 3)}$.

We prove that for $(\lambda, b)$ in regions I, II, or III there exist nonnegative matrices whose spectrum is $\{1, \lambda, bi - bi\}$. However, region IV is still unknown at the time of this submission.

#### 4.1.1 Solution of $\sigma = \{1, \lambda, bi, -bi\}$ when $\lambda = -1$ and $0 < b \leqslant 1$

A solution to this case which realizes regions I and II is relatively simple. Consider

the following matrix:

$$A = \frac{1}{2}\begin{bmatrix} 0 & 0 & 1+b & 1-b \\ 0 & 0 & 1-b & 1+b \\ 1-b & 1+b & 0 & 0 \\ 1+b & 1-b & 0 & 0 \end{bmatrix}.$$

Here $\sigma(A) = \{1, -1, bi, -bi\}$ for $0 < b \leqslant 1$. We prove this construction by using Theorem 6.

**Theorem 13.** *Let $\sigma = \{1, -1, ib, -ib\}$ be such that $0 \leqslant b \leqslant 1$. Then $\sigma$ is the spectrum of the nonnegative matrix $A$ where*

$$A = \frac{1}{2}\begin{bmatrix} 0 & 0 & 1+b & 1-b \\ 0 & 0 & 1-b & 1+b \\ 1-b & 1+b & 0 & 0 \\ 1+b & 1-b & 0 & 0 \end{bmatrix}.$$

*Proof.* Let

$$A_{1,2} = \frac{1}{2}\begin{bmatrix} 1+b & 1-b \\ 1-b & 1+b \end{bmatrix}$$

and

$$A_{2,1} = \frac{1}{2}\begin{bmatrix} 1-b & 1+b \\ 1+b & 1-b \end{bmatrix}.$$

Then

$$A_{1,2}A_{2,1} = \frac{1}{4}\begin{bmatrix} 1+b & 1-b \\ 1-b & 1+b \end{bmatrix}\begin{bmatrix} 1-b & 1+b \\ 1+b & 1-b \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 2(1+b)(1-b) & (1+b)^2+(1-b)^2 \\ (1+b)^2+(1-b)^2 & 2(1+b)(1-b) \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 2(1-b^2) & 2+2b^2 \\ 2+2b^2 & 2(1-b^2) \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1-b^2 & 1+b^2 \\ 1+b^2 & 1-b^2 \end{bmatrix}.$$

The characteristic polynomial of $A_{1,2}A_{2,1}$ is

$$\begin{vmatrix} \delta - \frac{1}{2}(1-b^2) & -\frac{1}{2}(1+b^2) \\ -\frac{1}{2}(1+b^2) & \delta - \frac{1}{2}(1-b^2) \end{vmatrix} = \left(\delta - \frac{1-b^2}{2}\right)^2 - \left(\frac{1+b^2}{2}\right)^2$$

$$= \delta^2 - \left(1-b^2\right)\delta - b^2$$

$$= (\delta - 1)\left(\delta + b^2\right).$$

Now, setting the characteristic polynomial equal to zero gives $\delta = 1$ or $\delta = -b^2$. But by Theorem 6, we know the eigenvalues of the matrix $A$ are just the square roots of $\delta$. This shows that the spectrum of $A$ is $\sigma = \{1, -1, bi, -bi\}$. $\square$

### 4.1.2 Solution of $\sigma = \{1, \lambda, bi, -bi\}$ when $0 < b \leqslant \frac{\sqrt{3}}{3}$ and $0 \leqslant \lambda \leqslant 1$

In this case $(\lambda, b)$ belongs to region III. An example of a nonnegative matrix in this region is:

$$A = \begin{bmatrix} \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} \\ 0 & \frac{2}{3} & 0 & \frac{1}{3} \end{bmatrix}$$

where $\sigma(A) = \{1, \frac{1}{3}, \frac{\sqrt{3}}{3}i, -\frac{\sqrt{3}}{3}i\}$.

Note that in general, the list $\{1, bi, -bi\}$ for $0 < b \leqslant \frac{\sqrt{3}}{3}$ satisfies Corollary 1. Hence there exists $A_1 \geqslant 0$ with spectrum $\{1, bi, -bi\}$. Since $\lambda \geqslant 0$ the nonnegative matrix $A = [\lambda] \oplus A_1$ is

$$A = \begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3}\left(\sqrt{3}b + 1\right) & \frac{1}{3}\left(1 - \sqrt{3}b\right) \\ 0 & \frac{1}{3}\left(1 - \sqrt{3}b\right) & \frac{1}{3} & \frac{1}{3}\left(1 + \sqrt{3}b\right) \\ 0 & \frac{1}{3}\left(1 + \sqrt{3}b\right) & \frac{1}{3}\left(1 - \sqrt{3}b\right) & \frac{1}{3} \end{bmatrix}$$

which realizes $\sigma$.

Therefore, a solution exists to the NIEP for $\sigma = \{1, \lambda, bi, -bi\}$ with Perron root 1 when $0 < b \leqslant \frac{\sqrt{3}}{3}$ and $0 \leqslant \lambda \leqslant 1$.

### 4.1.3 Solution of $\sigma = \{1, \lambda, bi, -bi\}$ when $0 < b \leqslant \frac{\sqrt{3}}{3}$ and $-1 < \lambda \leqslant 0$

Here we examine region II. We saw in Chapter 3 that Reams [29] completely solved the NIEP for a list of four numbers that have trace zero. The sufficient conditions

are $s_1 = 0$, $s_2 \geqslant 0$, $s_3 \geqslant 0$, and $4s_4 \geqslant s_2^2$. The solution due to Reams is

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{s_2}{4} & 0 & 1 & 0 \\ \frac{s_3}{4} & 0 & 0 & 1 \\ \frac{4s_4 - s_2^2}{16} & \frac{s_3}{12} & \frac{s_2}{4} & 0 \end{bmatrix}. \tag{2}$$

We will use this result to find the solution for region II. The solution is given in Theorem 14.

4.1.4 Solution of $\sigma = \{1, \lambda, bi, -bi\}$ when $\frac{\sqrt{3}}{3} \leqslant b \leqslant \sqrt{\frac{1}{8}(3\lambda^2 - 2\lambda + 3)}$ and $-1 < \lambda \leqslant 0$

Here we examine region I. Recall the upper limit for $b$ in region I comes from the JLL inequality

$$\begin{aligned} s_1^2 \leqslant 4s_2 &\Leftrightarrow (1+\lambda)^2 \leqslant 4\left(1 + \lambda^2 - 2b^2\right) \\ &\Leftrightarrow 1 + 2\lambda + \lambda^2 \leqslant 4 + 4\lambda^2 - 8b^2 \\ &\Leftrightarrow 0 \leqslant 3\lambda^2 - 2\lambda + \left(3 - 8b^2\right), \end{aligned}$$

which is a quadratic inequality in $b$ and $\lambda$ where the roots of $0 = 3\lambda^2 - 2\lambda + (3 - 8b^2)$ in terms of $b$ are $\lambda = \frac{1}{6}\left(2 \pm \sqrt{96b^2 - 32}\right)$. Now since $\lambda$ is a real number it follows that $96b^2 - 32 \geqslant 0 \Leftrightarrow 96b^2 \geqslant 32 \Leftrightarrow b^2 \geqslant \frac{1}{3} \Leftrightarrow b \geqslant \frac{\sqrt{3}}{3}$. Moreover, $\lambda = \frac{1}{3}$ is a double root to the quadratic equation when $b = \frac{\sqrt{3}}{3}$.

Now $b = \sqrt{\frac{1}{8}(3\lambda^2 - 2\lambda + 3)}$ is a root of the equation $b^2 = \frac{1}{8}(3\lambda^2 - 2\lambda + 3)$ from the JLL inequality. But $b$ is assumed to be a real number; so to verify this, we observe that $3\lambda^2 - 2\lambda + 3 > 0$ for all real $\lambda$ as the roots of this quadratic are complex numbers.

For $-1 < \lambda \leqslant 0$ and $\frac{\sqrt{3}}{3} \leqslant b \leqslant \sqrt{\frac{1}{8}(3\lambda^2 - 2\lambda + 3)}$ there exists a nonnegative matrix realizing $\sigma = \{1, \lambda, bi, -bi\}$ as its spectrum and we prove this in Theorem 14.

#### 4.1.5 Solution of $\sigma = \{1, \lambda, bi, -bi\}$ when $\frac{\sqrt{3}}{3} \leqslant b \leqslant \sqrt{\frac{1}{8}(3\lambda^2 - 2\lambda + 3)}$ and $0 \leqslant \lambda \leqslant 1$

This is region IV. Recall the upper limit for $b$ in region IV comes from the JLL inequality. For $0 \leqslant \lambda \leqslant \frac{1}{3}$ and $\frac{\sqrt{3}}{3} \leqslant b \leqslant \sqrt{\frac{1}{8}(3\lambda^2 - 2\lambda + 3)}$ Theorem 14 provides a nonnegative matrix that realizes $\sigma$. The remaining part of region IV when $\frac{1}{3} < \lambda \leqslant 1$ is still unresolved at the time of this submission.

#### 4.1.6 Results

The following theorem gives a nonnegative matrix realizing $\sigma = \{1, \lambda, bi, -bi\}$ when $(\lambda, b)$ belongs to regions I, II and parts of IV. Recall we completely solved region III in section (4.1.2). A part of this result is found in Reams [28].

**Theorem 14.** *Let $\sigma = \{1, \lambda, bi, -bi\}$ where $-1 < \lambda \leqslant 0$ and $0 < b \leqslant \sqrt{\frac{1}{8}(3\lambda^2 - 2\lambda + 3)}$ or $0 \leqslant \lambda \leqslant \frac{1}{3}$ and $\frac{\sqrt{3}}{3} \leqslant b \leqslant \sqrt{\frac{1}{8}(3\lambda^2 - 2\lambda + 3)}$. Suppose $\sigma$ satisfies all the necessary conditions of the NIEP. Then a nonnegative matrix $B$ which realizes $\sigma$ as its spectrum is $A + \frac{1+\lambda}{4}I$ where $A$ is Reams' solution* (2) *for the list $\{1 - \frac{1+\lambda}{4}, \lambda - \frac{1+\lambda}{4}, bi - \frac{1+\lambda}{4}, -bi - \frac{1+\lambda}{4}\}$ and $I$ is the identity matrix.*

*Proof.* Suppose $\{1, \lambda, bi, -bi\}$ can be realized by a nonnegative matrix $B$ with constant diagonal entries then $B - \frac{1+\lambda}{4}I$ has trace zero. A matrix with eigenvalues of

$B - \frac{1+\lambda}{4}I$, namely $\{1 - \frac{1+\lambda}{4}, \lambda - \frac{1+\lambda}{4}, bi - \frac{1+\lambda}{4}, -bi - \frac{1+\lambda}{4}\}$, is given by Reams as

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{s_2'}{4} & 0 & 1 & 0 \\ \frac{s_3'}{4} & 0 & 0 & 1 \\ \frac{4s_4' - \left(s_2'\right)^2}{16} & \frac{s_3'}{12} & \frac{s_2'}{4} & 0 \end{bmatrix},$$

where $s_i'$ are the moments corresponding to the eigenvalues of $B - \frac{1+\lambda}{4}I$. If we show $A \geqslant 0$ then it follows $A + \frac{1+\lambda}{4}I \geqslant 0$ and the spectrum of $A + \frac{1+\lambda}{4}I$ is the list $\{1, \lambda, bi, -bi\}$. We will analyze the entries of $A$.

Throughout, $s_i$ will denote the $i^{th}$ moment of the eigenvalues of $B$, namely $\sigma = \{1, \lambda, bi, -bi\}$. Note that we replace $1 + \lambda$ with $s_1$ in the following calculations. To show the $(2,1)$ and $(4,3)$ entries are nonnegative we need only show that $s_2' \geqslant 0$. Now,

$$\begin{aligned} s_2' &= trace\left(A^2\right) \\ &= trace\left(\left[B - \frac{1+\lambda}{4}I\right]^2\right) \\ &= trace\left(B^2 - \frac{1+\lambda}{2}B + \frac{(1+\lambda)^2}{16}I\right) \\ &= trace\left(B^2\right) - \frac{1+\lambda}{2}trace(B) + \frac{(1+\lambda)^2}{16}trace(I) \\ &= s_2 - \frac{1+\lambda}{2}s_1 + \frac{(1+\lambda)^2}{16} \cdot 4 \\ &= s_2 - \frac{1}{2}s_1^2 + \frac{1}{4}s_1^2 \\ &= s_2 - \frac{1}{4}s_1^2 \\ &= \frac{1}{4}\left(4s_2 - s_1^2\right) \end{aligned}$$

By assumption $\sigma$ satisfies the JLL conditions, so we have $4s_2 \geqslant s_1^2$. Thus $4s_2 - s_1^2 \geqslant 0$ from which follows the desired conclusion of $s_2' \geqslant 0$ for $-1 < \lambda \leqslant \frac{1}{3}$ and $0 < b \leqslant \sqrt{\frac{1}{8}\left(3\lambda^2 - 2\lambda + 3\right)}$.

To show the $(3,1)$ and $(4,2)$ entries are nonnegative we need to show that $s_3' \geqslant 0$. Now,

$$\begin{aligned} s_3' &= trace\left(A^3\right) \\ &= trace\left(\left[B - \frac{s_1}{4}I\right]^3\right) \\ &= trace\left(B^3 - 3\left(\frac{1+\lambda}{4}\right)B^2 + 3\left(\frac{1+\lambda}{4}\right)^2 B - \left(\frac{1+\lambda}{4}\right)^3 I\right) \\ &= \frac{8s_3 - 6s_1s_2 + s_1^3}{8}. \end{aligned}$$

Reams [29] shows that if $\sigma = \{r, \lambda_2, \lambda_3, \lambda_4\}$ where $r$ is the Perron root, then $8s_3 - 6s_1s_2 + s_1^3 = 3\left(r + \lambda_2 - \lambda_3 - \lambda_4\right)\left(r - \lambda_2 + \lambda_3 - \lambda_4\right)\left(r - \lambda_2 - \lambda_3 + \lambda_4\right)$. So in our case we have $r = 1$, $\lambda_2 = \lambda$, $\lambda_3 = bi$, and $\lambda_4 = -bi$. With substitution and recalling $1 + \lambda \geqslant 0$ from the Perron root condition, we have

$$\begin{aligned} 8s_3 - 6s_1s_2 + s_1^3 &= 3\left(1+\lambda\right)\left(1 - \lambda + 2bi\right)\left(1 - \lambda - 2bi\right) \\ &= 3\left(1+\lambda\right)\left[\left(1-\lambda\right)^2 + 4b^2\right] \\ &\geqslant 0. \end{aligned}$$

Thus $s_3' \geqslant 0$. Now to show the $(4,1)$ entry is nonnegative we will need the equivalent

of $s_4^{'}$ in terms of the moments of $B$.

$$\begin{aligned}
s_4^{'} &= trace\left(A^4\right) \\
&= trace\left(\left[B - \frac{1+\lambda}{4}I\right]^4\right) \\
&= trace\left(B^4 - 4\left(\frac{1+\lambda}{4}\right)B^3 + 6\left(\frac{1+\lambda}{4}\right)^2 B^2 - 4\left(\frac{1+\lambda}{4}\right)^3 B + \left(\frac{1+\lambda}{4}\right)^4 I\right) \\
&= s_4 - s_1 s_3 + \frac{3}{8}s_1^2 s_2 - \frac{s_1^4}{16} + \frac{s_1^4}{64} \\
&= s_4 - s_1 s_3 + \frac{3}{8}s_1^2 s_2 - \frac{3}{64}s_1^4.
\end{aligned}$$

Therefore

$$\begin{aligned}
4s_4^{'} - \left(s_2^{'}\right)^2 &= 4\left(s_4 - s_1 s_3 + \frac{3}{8}s_1^2 s_2 - \frac{3}{64}s_1^4\right) - \left(\frac{1}{4}\left(4s_2 - s_1^2\right)\right)^2 \\
&= \frac{16s_4 - 16s_1 s_3 + 8s_1^2 s_2 - 4s_2^2 - s_1^4}{4}.
\end{aligned}$$

Rewriting the numerator using $s_1$, $s_2$, $s_3$, and $s_4$ in terms of 1, $b$, and $\lambda$, we have

$$\begin{aligned}
&16s_4 - 16s_1 s_3 + 8s_1^2 s_2 - 4s_2^2 - s_1^4 \\
&= 16\left(1+\lambda^4+2b^4\right) - 16\left(1+\lambda\right)\left(1+\lambda^3\right) + 8\left(1+\lambda\right)^2\left(1+\lambda^2-2b^2\right) - \left(1+\lambda\right)^4 \\
&\quad - 4\left(1+\lambda^2-2b^2\right)^2 \\
&= 3 + 16b^4 - 4\lambda - 32b^2\lambda + 2\lambda^2 - 4\lambda^3 + 3\lambda^4 \\
&= 2\lambda^2 + 3\lambda^4 - \lambda\left(4 + 4\lambda^2 + 32b^2\right) + \left(3 + 16b^4\right)
\end{aligned}$$

which has positive terms with the only possible exception of $-\lambda\left(4+4\lambda^2+32b^2\right)$. But this term is nonnegative when $-1 < \lambda \leqslant 0$. Therefore, for $-1 < \lambda \leqslant 0$ and $0 \leqslant b \leqslant \sqrt{\frac{1}{8}\left(3\lambda^2 - 2\lambda + 3\right)}$ we have $4s_4^{'} - \left(s_2^{'}\right)^2 \geqslant 0$. Thus $B$ is nonnegative.

Assume $0 \leqslant \lambda \leqslant \frac{1}{3}$ and $\frac{\sqrt{3}}{3} \leqslant b \leqslant \sqrt{\frac{1}{8}\left(3\lambda^2 - 2\lambda + 3\right)}$. Consider from above, the polynomial $P\left(\lambda, b\right) = 3\lambda^4 - 4\lambda^3 + 2\lambda^2 - 4\left(8b^2+1\right)\lambda + \left(16b^4+3\right)$ over $0 \leqslant \lambda \leqslant \frac{1}{3}$

and $\frac{\sqrt{3}}{3} \leqslant b \leqslant \sqrt{\frac{3}{8}}$, a rectangle which contains the region $0 \leqslant \lambda \leqslant \frac{1}{3}$ and $\frac{\sqrt{3}}{3} \leqslant b \leqslant \sqrt{\frac{1}{8}\left(3\lambda^2 - 2\lambda + 3\right)}$. We will show $P(\lambda, b) \geqslant 0$ in this rectangle.

Now

$$\begin{aligned}\frac{dP}{d\lambda} &= 12\lambda^3 - 12\lambda^2 + 4\lambda - 4\left(8b^2 + 1\right) \\ &= 4\left[3\lambda^3 - 3\lambda^2 + \lambda - \left(8b^2 + 1\right)\right] \\ &\leqslant 4\left(3\lambda^3 - 3\lambda^2 + \lambda - \frac{11}{3}\right)\end{aligned}$$

using $b = \frac{\sqrt{3}}{3}$ and $\frac{d^2P}{d\lambda^2} = 36\lambda^2 - 24\lambda + 4 = 4\left(9\lambda^2 - 6\lambda + 1\right) = 4\left(3\lambda - 1\right)^2$, so $\frac{dP}{d\lambda} < 0$ on $\left[0, \frac{1}{3}\right] \mathrm{x} \left[\frac{\sqrt{3}}{3}, \sqrt{\frac{3}{8}}\right]$ and $\frac{d^2P}{d\lambda^2} \geqslant 0$.

Hence $P(\lambda, b)$ is decreasing and concave up for $(\lambda, b)$ in the rectangle. $\frac{d^2P}{d\lambda^2} = 0$ when $\lambda = \frac{1}{3}$ for $b$ in $\left[\frac{\sqrt{3}}{3}, \sqrt{\frac{3}{8}}\right]$. Note that $(\lambda, b) = \left(\frac{1}{3}, \frac{\sqrt{3}}{3}\right)$ and $(\lambda, b) = \left(0, \sqrt{\frac{3}{8}}\right)$ are points on the parabola coming from the JLL condition.

Now, $P\left(\frac{1}{3}, b\right) = 16b^4 - \frac{32}{3}b^2 + \frac{16}{9}$ with $P\left(\frac{1}{3}, \frac{\sqrt{3}}{3}\right) = 0$ and $P\left(\frac{1}{3}, \sqrt{\frac{3}{8}}\right) = \frac{1}{36}$. Consider $\frac{dP}{db}\left(\frac{1}{3}, b\right) = 64b\left(b - \frac{\sqrt{3}}{3}\right)\left(b + \frac{\sqrt{3}}{3}\right)$. This shows $\frac{dP}{db}\left(\frac{1}{3}, b\right) \geqslant 0$ if $\frac{\sqrt{3}}{3} \leqslant b \leqslant \sqrt{\frac{3}{8}}$. Clearly $P(0, b) \geqslant 0$ for $\frac{\sqrt{3}}{3} \leqslant b \leqslant \sqrt{\frac{3}{8}}$.

Since $P(\lambda, b)$ is decreasing and concave up with respect to $\lambda$ on $\left[0, \frac{1}{3}\right] \mathrm{x} \left[\frac{\sqrt{3}}{3}, \sqrt{\frac{3}{8}}\right]$ and $P(0, b) \geqslant 0$, $P\left(\frac{1}{3}, b\right) \geqslant 0$ for $\frac{\sqrt{3}}{3} \leqslant b \leqslant \sqrt{\frac{3}{8}}$, it follows that $P(\lambda, b) \geqslant 0$ for $(\lambda, b)$ in $\left[0, \frac{1}{3}\right] \mathrm{x} \left[\frac{\sqrt{3}}{3}, \sqrt{\frac{3}{8}}\right]$. Since this contains the region $0 \leqslant \lambda \leqslant \frac{1}{3}$ and $\frac{\sqrt{3}}{3} \leqslant b \leqslant \sqrt{\frac{1}{8}\left(3\lambda^2 - 2\lambda + 3\right)}$ it follows that $4s_4' \geqslant \left(s_2'\right)^2$.

□

Thus $A$ is nonnegative as claimed.

As an example, the following nonnegative matrix realizes $\{1, \frac{1}{6}, \sqrt{\frac{11}{32}}\, i, -\sqrt{\frac{11}{32}}\, i\}$ which is in the left portion of region IV.

$$\begin{bmatrix} \frac{7}{24} & 1 & 0 & 0 \\ 0 & \frac{7}{24} & 1 & 0 \\ \frac{1043}{4608} & 0 & \frac{7}{24} & 1 \\ \frac{4199}{110592} & \frac{1043}{13824} & 0 & \frac{7}{24} \end{bmatrix}$$

A final observation occurs when we attempt to use the following result in the unknown case. Here we can construct various real matrices having the spectrum $\{1, \lambda, bi, -bi\}$. This method did not always provide nonnegative matrices.

**Theorem 15.** [2] $\{1, \lambda_2, \ldots, \lambda_n\}$ *is realizable by a nonnegative matrix if and only if there exists some real matrix $B$ of order $n-1$ with spectrum $\{\lambda_2, \ldots, \lambda_n\}$ and some $(n-1)$-simplex $S \subset R^{n-1}$ with $0^{n-1} \in S$ such that $BS \subset S$.*

In our case $\{1, \lambda, bi, -bi\}$ is realizable by a nonnegative matrix if and only if there exists a 3 x 3 real matrix $B$ with spectrum $\{\lambda, bi, -bi\}$ and a tetrahedron $S$ containing $(0,0,0)$ such that point $p = (p_1, p_2, p_3) \in S \Rightarrow B\,[p_1, p_2, p_3]^T \in S$.

There are many choices for $B$ with spectrum $\{\lambda, bi, -bi\}$. One such matrix is

$$\begin{bmatrix} 0 & -b & 0 \\ b & 0 & 0 \\ 0 & 0 & \lambda \end{bmatrix}.$$

Now consider the tetrahedron with vertices $v_i = (x_i, y_i, z_i)$ for $i = 1, 2, 3, 4$. The

centroid of the tetrahedron with vertices $v_1, v_2, v_3, v_4$ is

$$\text{centroid} = \frac{1}{4}\left((x_1 + x_2 + x_3 + x_4), (y_1 + y_2 + y_3 + y_4), (z_1 + z_2 + z_3 + z_4)\right).$$

This centroid divides the line segment joining a vertex to the centroid of the opposite face in the ratio 3:1. The centroid of the face triangle is the intersection of all its medians.

Select $v_4 = (0, 0, 3)$. We desire the tetrahedron to be oriented so that the centroid is located at the origin. Then the other end of the line segment joining $v_4$ to the centroid is $(0, 0, -1)$. Note this point is the centroid of the triangle formed by $v_1$, $v_2$, and $v_3$.

Let the vertices be $v_1 = (x_1, 0, z_1)$, $v_2 = (x_2, y_2, z_2)$, $v_3 = (x_3, y_3, z_3)$, and $v_4 = (0, 0, 3)$. Thus the centroid of this tetrahedron is

$$\text{centroid} = \frac{1}{4}\left((x_1 + x_2 + x_3), (y_2 + y_3), (z_1 + z_2 + z_3 + 3)\right).$$

This gives $v_3 = (-x_1 - x_2, -y_2, -z_1 - z_2 - 3)$.

To use Theorem 15 we consider the real matrix

$$B = \begin{bmatrix} 0 & -b & 0 \\ b & 0 & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

where we are rotating a tetrahedron by 90 degrees and shrinking it by $b$ relative to the $xy$-plane and then compressing it by $\lambda$ along the $z$-axis. We want to know if every point in the tetrahedron when acted on by $B$ will remain in the original tetrahedron.

As a tetrahedron is a convex set, any point $p = (p_1, p_2, p_3)$ in the tetrahedron can be described by the convex sum $p = \sum_{i=1}^{4} t_i v_i$ where $t_i \geqslant 0$ for $i = 1, 2, 3, 4$ and $\sum_{i=1}^{4} t_i = 1$.

We can then test to see if there exists a nonnegative matrix associated with $\sigma = \{1, \lambda, -bi, bi\}$ by solving the following system of linear equations for the coefficients $t_1, t_2, t_3, t_4$:

$$\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{bmatrix} = \begin{bmatrix} -bp_2 \\ bp_1 \\ \lambda p_3 \\ 1 \end{bmatrix}.$$

Here $p = (p_1, p_2, p_3)$ belonged to the original tetrahedron.

Now the matrix

$$A = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

is nonsingular by construction as the vertices are noncoplanar; therefore, we may find its inverse to solve for the coefficients.

Since the extreme points of the tetrahedron are its vertices, if the vertices when acted upon by $B$ remain part of the original tetrahedron, then every point of the tetrahedron when acted on by $B$ will remain part of the original tetrahedron. Therefore we need to test those four vertices, $v_i$ for $i = 1, 2, 3, 4$, by solving for their correspond-

ing coefficients. If these coefficients remain nonnegative then the matrix $C$ whose columns are the coefficients corresponding to $v_i$ for $i = 1, 2, 3, 4$ will be a nonnegative matrix realizing $\sigma$ using Theorem 15.

Consider the following example. Let $v_1 = (4, 0, -4)$ and $v_2 = (-1, 5, -1)$. Then $v_3 = (-3, -5, 2)$ while $v_4 = (0, 0, 3)$. Solving this problem with *Mathematica* when $\lambda = \frac{2}{3}$ and $b = 0.6$, we find the matrix $C$ is approximately

$$C = \begin{bmatrix} 0.3622 & 0.0206 & 0.5339 & 0.0833 \\ 0.7222 & 0.6806 & -0.4861 & 0.0833 \\ 0.2422 & 0.8001 & -0.1261 & 0.0833 \\ -0.3267 & -0.5017 & -0.1261 & 0.75 \end{bmatrix}.$$

Notice this attempt fails to take $v_1, v_2, v_3$ into the original tetrahedron as each convex sum coefficient fails to be nonnegative, but the matrix $C$ does indeed have spectrum $\{1, \frac{2}{3}, 0.6i, -0.6i\}$.

### 4.2 NIEP for $\sigma = \{1, \lambda, a + bi, a - bi\}$

Let us now consider the following four cases relative to the complete list of four numbers where $a \neq 0$.

1. Consider $\lambda = 1$ and $a^2 + b^2 < 1$. Since the Perron root is repeated any nonnegative matrix realizing $\sigma$ must be reducible. By the sufficient conditions for $\{1, a + bi, a - bi\}$ to be realizable by a nonnegative matrix from Theorem 9. They are

    (a) $1 + 2a \geqslant 0$, the trace is nonnegative, and

(b) $(1+2a)^2 \leqslant 3\,(1+2a^2-2b^2)$, from the JLL condition.

If (a) and (b) are satisfied, then by Theorem 9 there exists a nonnegative matrix, $A'$, realizing $\{1, a+bi, a-bi\}$. Using this matrix, we obtain the following reducible nonnegative matrix having spectrum $\sigma$.

$$A = \begin{bmatrix} A' & 0 \\ 0 & 1 \end{bmatrix}.$$

2. Suppose $\lambda = -1$ and $a^2+b^2 < 1$. Then $\sigma = \{1, -1, a+bi, a-bi\}$, and for the trace condition to hold we need $a \geqslant 0$. Should a nonnegative matrix exist having such a spectrum, it could not be reducible as no partition would work with the $\lambda = -1$. Therefore, the only possible nonnegative matrix must be irreducible.

   Since $|-1| = 1$ we have $S = \{1, -1\}$ in Theorem 3. This means $a+bi$ and $a-bi$ are on the imaginary axis giving $a = 0$. Since $a^2+b^2 < 1$ we get $-1 < b < 1$. But we have seen solutions to $\{1, -1, bi, -bi\}$ for $0 < b < 1$ in subsection (4.1.1).

3. $\lambda = -1$ and $a^2+b^2 = 1$. Then we have four eigenvalues of the same modulus and they must occur at $1$, $-1$, $i$, $-i$. Therefore, the only possible spectrum with these conditions that have a nonnegative matrix associated with it is $\sigma = \{1, -1, i, -i\}$. We have seen such a matrix in subsection (4.1.1).

4. Suppose $\lambda = 1$ and $a^2+b^2 = 1$. By the Perron-Froebenius Theorem we know a repeating Perron root requires the nonnegative matrix with spectrum $\sigma$ to be

reducible. Therefore the only possibility is when $a \pm bi$ is in the closed triangular region with vertices $1, e^{\frac{2\pi i}{3}}, e^{-\frac{2\pi i}{3}}$ and thus $\{1, a+bi, a-bi\}$ is the spectrum of some nonnegative 3 x 3 matrix from Corollary 1, call it $B$. Then

$$A = \begin{bmatrix} B & 0 \\ 0 & 1 \end{bmatrix}$$

realizes $\sigma = \{1, 1, a+bi, a-bi\}$.

5. Suppose $|\lambda| < 1$ and $a^2 + b^2 = 1$. If we consider an irreducible nonnegative matrix then we must abide by Theorem 3 which says that if we have three values with modulus one, the remaining value $\lambda$ must have two other distinct numbers in $\sigma$ with the same modulus as $\lambda$. That is not possible. When we consider the reducible nonnegative matrix, then $a \pm bi$ must be in the closed triangular region with vertices $1, e^{\frac{2\pi i}{3}}, e^{-\frac{2\pi i}{3}}$ and $\lambda \geqslant 0$. In this case, Corollary 1 guarantees $B \geqslant 0$ with spectrum $\{1, a+bi, a-bi\}$. Then

$$A = \begin{bmatrix} B & 0 \\ 0 & \lambda \end{bmatrix}$$

has spectrum $\sigma$.

6. Suppose $|\lambda| < 1$ and $a^2 + b^2 < 1$. This case is still unresolved.

As a final result of this chapter we make the following observation.

The example below shows that it is not possible to solve the 4 x 4 NIEP by considering only matrices whose main diagonal elements are equal as in Theorem 14.

Suppose $\sigma = \{r, \lambda, a+bi, a-bi\}$ has such a solution, then Reams' expression for $s_3 = 3\left(r+\lambda_2-\lambda_3-\lambda_4\right)\left(r-\lambda_2+\lambda_3-\lambda_4\right)\left(r-\lambda_2-\lambda_3+\lambda_4\right)$ gives $s_3 = 3\left(r+\lambda-2a\right)\left(r-\lambda+2bi\right)\left(r-\lambda-2bi\right) \geqslant 0$. This implies $\left(r+\lambda-2a\right) \geqslant 0$. But for the following matrix from [28]

$$A = \begin{bmatrix} 0 & 6 & 0 & 0 \\ 1 & \frac{11}{3} & \frac{2}{3}+\frac{1}{\sqrt{3}} & \frac{2}{3}-\frac{1}{\sqrt{3}} \\ 1 & \frac{2}{3}-\frac{1}{\sqrt{3}} & \frac{11}{3} & \frac{2}{3}+\frac{1}{\sqrt{3}} \\ 1 & \frac{2}{3}+\frac{1}{\sqrt{3}} & \frac{2}{3}-\frac{1}{\sqrt{3}} & \frac{11}{3} \end{bmatrix}$$

the spectrum is $\sigma(A) = \{6, -1, 3+i, 3-i\}$. Here $\left(r+\lambda-2a\right) = 6-1-6 = -1 < 0$. Hence $\{6, -1, 3+i, 3-i\}$ cannot be realized by a matrix $B \geqslant 0$ with equal entries along the main diagonal.

## 4.3 Conclusion

In this chapter we have identified nonnegative matrices realizing a list of four complex numbers. In the case where $\sigma = \{1, \lambda, bi, -bi\}$, the main result is Theorem 14 which summarizes the known regions. A part of region IV is still unresolved. In the general case we addressed the boundary point possibilities leaving open the case when $\sigma = \{1, \lambda, a+bi, a-bi\}$ where $|\lambda| < 1$ and $a^2+b^2 < 1$.

We also have a construction technique to find matrices having certain lists as spectrums using Theorem 15 and *Mathematica*. Although this process was not fruitful in providing a nonnegative matrix, it gave the opportunity to test many different real matrices.

CHAPTER 5

# Real versus Symmetric Nonnegative Inverse Eigenvalue Problem

The general NIEP is open. Some progress has been made in the problem of the existence of a nonnegative matrix with a prescribed real spectrum. Chapter 5 focuses on lists of real numbers.

## 5.1 Real Nonnegative Inverse Eigenvalue Problem (RNIEP)

The RNIEP is a subproblem of the NIEP with one additional constraint. The RNIEP restricts the elements of $\sigma = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ to be from the set of real numbers. *The RNIEP is to determine the necessary and sufficient conditions on $\sigma = \{\lambda_1, \lambda_2, \ldots, \lambda_n\} \subset R$ so that there exists a nonnegative matrix having $\sigma$ as its spectrum.*

As we have seen in Chapter 3, Suleimanova [32] gave a sufficient condition for the existence of a nonnegative matrix when the list contains real numbers. The simple proof given here for Suleimanova's Theorem is due to Perfect [24].

**Theorem 16.** [22] *Let $\sigma = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ be a list of $n$ real numbers satisfying*

$$\lambda_1 + \lambda_2 + \cdots + \lambda_n \geqslant 0$$

$$\lambda_j < 0, \textit{ for } j = 2, 3, \ldots, n.$$

*Then, there exists a nonnegative* $n \times n$ *matrix with spectrum* $\sigma$.

*Proof.* Let

$$f(\lambda) = \prod_{j=1}^{n} (\lambda - \lambda_j) = \lambda^n - \sum_{j=1}^{n} c_j \lambda^{n-j}.$$

We assert that all $c_2, c_3, \ldots c_n$ must be positive. For the polynomial $f(-\lambda) = (-1)^n \left[ \lambda^n + \sum_{j=1}^{n} (-1)^{j-1} c_j \lambda^{n-j} \right]$ has exactly $(n-1)$ positive roots. Therefore, by Descartes' rule of signs, the number of variations of sign in the sequence

$$1, c_1, -c_2, c_3, -c_4, \ldots, (-1)^{n-1} c_n$$

is exactly $(n-1)$. However, $c_1 = \lambda_1 + \lambda_2 + \cdots + \lambda_n$ is nonnegative, and it follows that all the other $c_j$ must be positive. Hence the companion matrix of $f(\lambda)$,

$$\begin{bmatrix} 0 & 1 & 0 & & & 0 & 0 \\ 0 & 0 & 1 & & & 0 & 0 \\ \vdots & & & \ddots & \ddots & & \vdots \\ 0 & & & & & 0 & 1 \\ c_n & c_{n-1} & c_{n-2} & & \cdots & c_2 & c_1 \end{bmatrix}$$

is nonnegative and has eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$.

□

The following corollaries are due to Perfect [24].

**Corollary 2.** [22] *Let* $\sigma = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$ *be a list of* $n$ *real numbers where*

$$\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_p \geqslant 0 > \lambda_{p+1} \geqslant \cdots \geqslant \lambda_n \text{ and}$$

$$\lambda_1 + \lambda_{p+1} + \cdots + \lambda_n \geqslant 0.$$

*Proof.* Use the proof of Theorem 16 to construct a nonnegative $(n-p+1)$ x $(n-p+1)$ matrix $B$ with eigenvalues $\lambda_1, \lambda_{p+1}, \lambda_{p+2}, \dots, \lambda_n$. Then the nonnegative matrix $B \oplus diag\,(\lambda_2, \dots, \lambda_p)$ has spectrum $\sigma$. □

**Corollary 3.** [22] *Let* $\sigma$ *be a list of* $n$ *real numbers, and suppose that it is possible to partition* $\sigma$ *into sublists in such a way that (i) each sublist contains one or more nonnegative numbers and (ii) the sum of the negative numbers (if any) in each sublist does not exceed the largest positive number in the sublist. Then* $\sigma$ *is the spectrum of a nonnegative matrix.*

The proof of this corollary follows from applying Corollary 2 to each of the sublists and taking a direct sum of nonnegative matrices.

Suleimanova also presented the following conjecture in 1965 regarding the RNIEP. Conjecture: If the numbers $1, \lambda_2, \dots, \lambda_n$ satisfy $1 + \sum_{i=2}^{n} \lambda_i^k > 0$, for $k = 1, 2, 3, \dots,$ and $|\lambda_i| < 1$ for $i = 2, 3, \dots, n$, then the numbers are eigenvalues of some stochastic matrix.

This conjecture is not satisfied for $n \geqslant 5$ as the list $\{1, 1, -\frac{2}{3}, -\frac{2}{3}, -\frac{2}{3}\}$ with repeated Perron root, satisfies the hypotheses but cannot be the spectrum of a nonnegative matrix as seen in Chapter 3. Although this conjecture is not true, it was Suleimanova's work which inspired many new results in the RNIEP. One improvement to Suleimanova's condition comes from Salzmann [30].

**Theorem 17 (Salzmann's Theorem).** [30] *Let $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$ be real numbers such that*

1. $\frac{1}{2}(\lambda_i + \lambda_{n-i+1}) \leqslant \frac{1}{n} \sum\limits_{j=1}^{n} \lambda_j$ *when* $i = 2,\ 3,\ \ldots,\ \left[\frac{n+1}{2}\right]$, *and*
2. $\sum\limits_{i=1}^{n} \lambda_i \geqslant 0$.

*Then there exists an $n$ x $n$ nonnegative diagonalizable matrix having $\sigma = \{\lambda_1,\ \lambda_2,\ \ldots,\ \lambda_n\}$ as its spectrum. Furthermore if the inequalities (1) and (2) are strict, then there exists a positive diagonalizable matrix with spectrum $\sigma$.*

Salzmann's condition clearly extends Suleimanova's condition. Moreover, if we consider $\sigma = \{7, 2, -2, -3, -3\}$, by Corollary 3, there is a nonnegative reducible matrix $A$ with spectrum $\sigma$. However Salzmann's theorem shows that $A$ must be similar to a positive and thus an irreducible matrix.

Kellogg [15] produced a set of sufficient conditions to solve the RNIEP. He noticed that if there exists a real matrix $A$ having spectrum $\sigma = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ such that $\lambda_1 > |\lambda_j|$ for $2 \leqslant j \leqslant n$, then $A$ is similar to a positive matrix provided that $\sigma$'s

elements meet certain conditions. Theorem 18 contains Kellogg's sufficient conditions for a real matrix having spectrum $\sigma$ to be similar to a positive matrix.

**Theorem 18.** [15] *Let $A$ be a real matrix having real eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that $\lambda_1 > \lambda_2 \geqslant \cdots \geqslant \lambda_n$ and $\lambda_1 > |\lambda_n|$. Let $r$ be the greatest integer with $\lambda_r \geqslant 0$. Define $\delta_i = \lambda_{n+2-i}$ for $i = 2, \ldots, s = n - r + 1$ and*
*$K = \{i \in \{2, \ldots, \min\{r, s\}\} : \lambda_i + \delta_i < 0\}$. If $\lambda_1 + \sum_{i \in K, i<h} (\lambda_i + \delta_i) + \delta_h > 0$ for all $h \in K$ and if $\lambda_1 + \sum_{i \in K} (\lambda_i + \delta_i) + \sum_{j=r+1}^{s} \delta_j > 0$ then $A$ is similar to a positive matrix.*

Kellogg proved Theorem 18 by using the following ideas. Since $A$ is a real matrix having real eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that $\lambda_1 > \lambda_2 \geqslant \cdots \geqslant \lambda_n$ and $\lambda_1 > |\lambda_n|$, then $A$ is similar to a strictly positive matrix if and only if there is an $(n-1)$-simplex $S$ in the set of vectors orthogonal to $A^T$'s eigenspace for $\lambda_1$ such that $AS \subset \lambda_1 S_0$. Here $S_0$ is defined to be the interior of $S$ where $S_0$ excludes any points of $S$ that require a zero coefficient on any of the $n-1$ vertices of $S$ in writing the convex sum to obtain the points. Kellogg produced a real $(n-1)$ x $(n-1)$ matrix $A$ having the normalized eigenvalues $1, \frac{\lambda_2}{\lambda_1}, \ldots, \frac{\lambda_n}{\lambda_1}$ and showed that $A$ takes $S$ into $S_0$.

It was Fiedler [7] who observed that changing the strict inequalities in Kellogg's conditions allowed Theorem 18 to apply to nonnegative matrices. In Theorem 19, Fiedler gives the extenstion of Theorem 18 for the nonnegative case. We call this as Kellogg's Theorem.

**Theorem 19 (Kellogg's Theorem).** [7] *Let $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$, $\lambda_1 \geqslant |\lambda_n|$, and let $r$ be the greatest integer with $\lambda_r \geqslant 0$. Let $\delta_i = \lambda_{n+2-i}$ for $i = 2, \ldots, s = n - r + 1$. Define $K = \{i \in \{2, \ldots, \min\{r, s\}\} : \lambda_i + \delta_i < 0\}$. If $\lambda_1 + \sum\limits_{i \in K, i<h} (\lambda_i + \delta_i) + \delta_h \geqslant 0$ for all $h \in K$ and if $\lambda_1 + \sum\limits_{i \in K} (\lambda_i + \delta_i) + \sum\limits_{j=r+1}^{s} \delta_j \geqslant 0$ then $\sigma = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ is the spectrum of some $n$ x $n$ nonnegative matrix.*

In the following, Fiedler [7] also gives sufficient conditions for a set of real numbers to be the spectrum of a nonnegative matrix.

**Theorem 20 (Fiedler's Theorem).** [7] *If $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$ and $\lambda_1 + \lambda_n + \sum\limits_{j=1}^{n} \lambda_j \geqslant \frac{1}{2} \sum\limits_{i=2}^{n-1} |\lambda_i + \lambda_{n+1-i}|$, then $\sigma = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ is the spectrum of some $n$ x $n$ nonnegative matrix.*

Fiedler proved that if the conditions of Salzmann's Theorem are satisfied, then the conditions of Fiedler's Theorem are satisfied; moreover, if the conditions of Fiedler's Theorem are satisfied then the conditions of Kellogg's Theorem are satisfied. That is Salzmann $\Rightarrow$ Fiedler $\Rightarrow$ Kellogg. None of the converses are true. To see this consider $\sigma_1 = \{5, 4, -2, -4\}$ and $\sigma_2 = \{3, 2, -1, -1, -3\}$. Here notice that $\sigma_1$ satisfies Fiedler's condition but not Salzmann's and that $\sigma_2$ satisfies Kellogg's condition but not Fiedler's. Therefore the most general sufficient condition from the above theorems comes from Kellogg.

But even Kellogg's Theorem has its limitations. Notice that the set $\sigma = \{1, 0.6, -0.3, -0.35, -0.45, -0.5\}$ does not satisfy Kellogg's Theorem, and consequently none of the prior theorems can identify $\sigma$ is a spectrum of some nonnegative

matrix. As it turns out, $\sigma$ is the spectrum of a 6 x 6 nonnegative matrix. Borobia proved this. Borobia [2] was able to extended Kellogg's conditions in Theorem 21.

**Theorem 21 (Borobia's Theorem).** [2] *Let $\sigma = \{\lambda_1,\ \lambda_2,\ \ldots,\ \lambda_n\}$ be a set of real numbers with $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$. Let $r$ be the greatest integer with $\lambda_r \geqslant 0$. If some partition $J_1 \cup \cdots \cup J_s$ of $J = \{\lambda_{r+1},\ \ldots,\ \lambda_n\}$ exists such that $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_r \geqslant \sum_{\lambda \in J_s} \lambda \geqslant \sum_{\lambda \in J_{s-1}} \lambda \geqslant \cdots \geqslant \sum_{\lambda \in J_1} \lambda$ satisfies Kellogg's Theorem, then $\sigma$ is the spectrum of some $n$ x $n$ nonnegative matrix.*

To see an application of Borobia's Theorem we return to the example $\sigma = \{1, 0.6, -0.3, -0.35, -0.45, -0.5\}$. Kellogg's Theorem fails for this $\sigma$ because here we have $K = \emptyset$ as $0.6 + (-0.5) \geqslant 0$ but $1 + (-0.3) + (-0.35) + (-0.45) \ngeqslant 0$. Therefore we partition $\sigma$ in the following way: $\{1, 0.6, (-0.3) + (-0.35), (-0.45) + (-0.5)\}$. Now applying Kellogg's Theorem to $\{1, 0.6, -0.65, -0.95\}$, we have $K = \{2\}$ as $0.6 + (-0.95) < 0$. So checking the remaining conditions we have $1 + (-0.95) \geqslant 0$ and $1 + 0.6 + (-0.65) + (-0.95) \geqslant 0$; therefore, there exists a nonnegative matrix realizing $\sigma$ by Borobia's Theorem.

## 5.2 Symmetric Nonnegative Inverse Eigenvalue Problem (SNIEP)

One additional restriction to the RNIEP gives the SNIEP. *The SNIEP requires finding necessary and sufficient conditions on $\sigma = \{\lambda_1,\ \lambda_2,\ \ldots,\ \lambda_n\} \subset R$ so that there exists an $n$ x $n$ symmetric nonnegative matrix having $\sigma$ as its spectrum.*

Fiedler [7] proved that the conditions of Kellogg's Theorem are also sufficient for

the existence of a symmetric nonnegative matrix. This result is in Theorem 22.

**Theorem 22.** [7] *Let* $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$ *satisfy the conditions of Kellogg's Theorem. Then there exists an* $n \times n$ *symmetric nonnegative matrix with eigenvalues* $\lambda_1, \lambda_2, \ldots, \lambda_n$.

Soules [31] gives a sufficient condition using the all one's vector as the Perron vector for the existence of a symmetric nonnegative matrix having real $\sigma$ as its spectrum. His conditions are in Theorem 23. We will discuss Soules construction further in Chapter 6.

**Theorem 23.** [31] *Let* $[1, 1, \ldots, 1]^T$ *be the Perron vector corresponding to the Perron root* $\lambda_1$. *Let* $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$ *and* $\frac{1}{n+2}\lambda_1 + \frac{n+1-m}{(n+2)(m+1)}\lambda_2 + \sum_{k=1}^{m} \frac{\lambda_{n-2k+2}}{(k+1)k} \geqslant 0$ *where* $m = \left[\frac{n+1}{2}\right]$, *then there exists an* $n \times n$ *symmetric, doubly stochastic matrix* $A$ *such that* $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ *is the spectrum of* $\lambda_1 A$.

Radwan [27] shows that the conditions of Theorem 22 and the conditions of Theorem 23 are not comparable. For example, $\sigma_1 = \{5, 3, -2, -2, -2, -2\}$ satisfies Theorem 23 but not Theorem 22, and $\sigma_2 = \{5, 3, -2, -2, -4\}$ satisfies Theorem 22 but not Theorem 23. Radwan also points out that Theorem 22 does not work for $\sigma = \{9, 6, -2, -4, -4, -4\}$; however, the following Theorem 24 does.

**Theorem 24 (Loewy's Theorem).** [27] *Let* $\lambda_1 \geqslant \lambda_2 > 0 \geqslant \lambda_3 \geqslant \cdots \geqslant \lambda_n$ *be real numbers. If for some partition* $J_1 \cup J_2$ *of* $J = \{-\lambda_3, -\lambda_4, \ldots, -\lambda_n\}$, *we have* $\lambda_1 \geqslant \sum_{\lambda \in J_1} \lambda \geqslant \sum_{\lambda \in J_2} \lambda$, *and* $\sum_{j=1}^{n} \lambda_j \geqslant 0$, *then* $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ *is the spectrum of some* $n \times n$ *symmetric nonnegative matrix.*

As Loewy's Theorem is applicable to a $\sigma$ containing exactly two positive elements, the question arises as to whether or not a general partitioning can be used for the SNIEP. Here is where a result similar to Borobia's Theorem for the RNIEP would be useful in the SNIEP. Radwan answers this question. Radwan [27] proved that the conditions of Borobia's Theorem are also sufficient for the existence of a symmetric nonnegative matrix. This result is in the following Theorem 25.

**Theorem 25 (Radwan's Theorem).** [27] *Let $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_M \geqslant 0$ be nonnegative numbers and $\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_N > 0$ be positive numbers. Let $S$ be an integer such that $1 \leqslant S \leqslant N$. Let $J_1 \cup J_2 \cup \cdots \cup J_s$ be an $S$-partition of $J = \{\mu_1,\ \mu_2, \ldots,\ \mu_N\}$ where $\sum_{\mu \in J_1} \mu \geqslant \sum_{\mu \in J_2} \mu \geqslant \cdots \geqslant \sum_{\mu \in J_S} \mu$. Define $T_j = \sum_{\mu \in J_j} \mu$ for $j = 1,\ 2,\ \ldots,\ S$. If $\sigma_S = \{\lambda_1,\ \lambda_2,\ \ldots,\ \lambda_M,\ -T_S,\ -T_{S-1},\ \ldots,\ -T_1\}$ satisfies the hypotheses of Kellogg's Theorem, then $\sigma = \{\lambda_1,\ \lambda_2,\ \ldots,\ \lambda_M,\ -\mu_N,\ -\mu_{N-1},\ \ldots,\ -\mu_1\}$ is the spectrum of some $(M+N+1)$ x $(M+N+1)$ symmetric nonnegative matrix.*

A nice result due to Fiedler helps in the construction of symmetric nonnegative matrices when spectra of two symmetric nonnegative matrices are known. In [7], we find the following two theorems to be very useful in this construction. We use $S_n$ to denote the set of lists of length $n$ realizable by an $n$ x $n$ symmetric nonnegative matrix.

**Theorem 26 (Fiedler's Symmetric Theorem).** [7] *If $\{\alpha_1, \alpha_2, \ldots, \alpha_m\} \in S_m$ with $\alpha_1$ as the Perron root, $\{\beta_1, \beta_2, \ldots, \beta_n\} \in S_n$ with $\beta_1$ as the Perron root, and $\alpha_1 \geqslant \beta_1$, then for any $\sigma \geqslant 0$, $\{\alpha_1 + \sigma, \beta_1 - \sigma, \alpha_2, \ldots, \alpha_m, \beta_2, \ldots, \beta_n\} \in S_{m+n}$.*

**Theorem 27.** [7] *Let $A$ be a symmetric $m$ x $m$ matrix with eigenvalues $\alpha_1, \ldots, \alpha_m$ and let $u$, $\|u\|_2 = 1$, be a unit eigenvector corresponding to $\alpha_1$; let $B$ be a symmetric $n$ x $n$ matrix with eigenvalues $\beta_1, \ldots, \beta_n$ and let $v$, $\|v\|_2 = 1$, be a unit eigenvector corresponding to $\beta_1$. Then for any $\rho$, the matrix*

$$C = \begin{bmatrix} A & \rho u v^T \\ \rho v u^T & B \end{bmatrix}$$

*has eigenvalues $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n, \gamma_1, \gamma_2$, where $\gamma_1, \gamma_2$ are eigenvalues of the matrix*

$$C = \begin{bmatrix} \alpha_1 & \rho \\ \rho & \beta_1 \end{bmatrix}.$$

## 5.3 The RNIEP and the SNIEP are Different

The RNIEP and the SNIEP are not equivalent problems. In the case when $\sigma = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ is real we consider both the RNIEP and the SNIEP. We will restate the question:

1. the real nonnegative inverse eigenvalue problem (RNIEP) asks which lists of $n$ real numbers occur as the spectrum of a nonnegative matrix $A \in M_n$, and

2. the symmetric nonnegative inverse eigenvalue problem (SNIEP) asks which lists of $n$ real numbers occur as the spectrum of a symmetric nonnegative matrix $A \in M_n$.

Obviously, if the SNIEP has a solution for a given $\sigma$ then the RNIEP also has a solution. In particular, in the case $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n \geqslant 0$ then both problems

trivially have a solution, namely the diagonal matrix $D = diag\,(\lambda_1, \lambda_2, \ldots, \lambda_n)$. When at least one of the $\lambda$'s is negative, are these problems equivalent? In other words, if there is a solution for the RNIEP does it imply there is a solution for the SNIEP? This question was raised by Hershkowitz [10] in 1978.

We have seen several sufficient conditions for the RNIEP in the previous sections. We also have seen that all those conditions actually solve the SNIEP. But in 1996, Johnson, Laffey, and Loewy showed that the two problems are different.

The following is a brief description of the result of Johnson, Laffey, and Loewy [13]. According to Boyle and Handelman [3] if $\lambda_1, \ldots, \lambda_n$ meet certain conditions, it is possible to append sufficiently many 0's: $\lambda_{n+1} = \lambda_{n+2} = \cdots = \lambda_m = 0$ so that $\lambda_1, \ldots, \lambda_n, \lambda_{n+1}, \lambda_{n+2}, \cdots, \lambda_m$ are the eigenvalues of a positive matrix $A \in M_n$. However, $m$ and rank $A$ may have to be large relative to $n$. It is possible to give examples for which $m$ must be very large. If $A$ were a symmetric nonnegative matrix, then rank $A$ would be very low relative to $m$. In [13], it is shown that if $A \in M_m$ is a sufficiently low rank nonnegative matrix, then there is another nonnegative matrix $A' \in M_{m'}$ with $m' < m$ such that the nonzero part of the spectrum of $A'$ agrees with that of $A$. This rules out the possibility of the SNIEP having a solution for certain lists for which the Boyle and Handelman result guarantees a solution to the RNIEP.

As a specific example, consider $n = 6$ and $\lambda_1 = \sqrt[3]{51} + \epsilon$, $\lambda_2 = \lambda_3 = \lambda_4 = 1$, and $\lambda_5 = \lambda_6 = -3$ for small $\epsilon > 0$. For $\epsilon > 0$, $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6\}$ satisfies the Boyle and Handelman conditions. For each $m > 6$, there is an $\epsilon_m > 0$ such that at least

$m-6$ 0's must be added to obtain the nonnegative realization $A \in M_m$ guaranteed by Boyle and Handelman. Consider $m = 22$. For some $\epsilon_{22} > 0$, Johnson, Laffey and Loewy show that there is no symmetric nonnegative matrix of any order whose nonzero spectrum is $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6$.

In Chapter 6, we will discuss the results that show for lists of length $n \leqslant 4$ the RNIEP and the SNIEP are equivalent. But knowing that the RNIEP and the SNIEP are different in general, what is the smallest list for which they are different? We will show in Chapter 6 that the RNIEP and the SNIEP are different for lists of length five.

# CHAPTER 6

## Lists of Length Five

In this chapter we derive new results concerning the RNIEP and the SNIEP. Recall that the RNIEP is to determine the necessary and sufficient conditions for a list of $n$ real numbers, $\sigma = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, to be the spectrum of some nonnegative matrix. The SNIEP is similar with the only difference being that $\sigma$ must now be the spectrum of some symmetric nonnegative matrix. In this chapter we will consider the values of $\sigma$ to always be ordered such that $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$ where $\lambda_1 \geqslant |\lambda_n|$. In other words, $\lambda_1$ will be the Perron root should a nonnegative matrix exist. Of course we already have seen some necessary conditions for a real $\sigma$ such as the moments must be nonnegative, the Perron root must be a part of $\sigma$, and the JLL condition must hold. These conditions are sufficient in lists of four or fewer real numbers and we will see this in the section (6.2). These conditions are not sufficient in lists of five or more real numbers, and we will see this in section (6.2). We begin by stating some recent results concerning the RNIEP and the SNIEP in the case where $n \leqslant 4$ in section (6.1). It is for these smaller dimensions that the problems are equivalent.

## 6.1 The Soules Approach to the SNIEP

Two recent papers, McDonald and Neumann [21] and Knudsen and McDonald [16], discuss the Soules approach to the SNIEP for $n \leqslant 5$. A list of $n$ real numbers, $\{1, \lambda_2, \ldots, \lambda_n\}$ satisfying the conditions $1 \geqslant |\lambda_i|$ for $i = 2, 3, \ldots, n$ and $1 + \lambda_2 + \cdots + \lambda_n \geqslant 0$ define a polytope in $R^n$ called the trace nonnegative polytope denoted by $\tau_n$. Any list of $n$ real numbers realizable as the spectrum of a symmetric nonnegative matrix must be in this polytope $\tau_n$, however for $n \geqslant 5$, $\tau_n$ contains points which are not spectrum of any symmetric nonnegative matrices. For example, $\{1, 1, -\frac{2}{3}, -\frac{2}{3}, -\frac{2}{3}\}$ and $\{1, \frac{1}{2}, \frac{1}{2}, -1 - 1\}$ are not the spectrum of any nonnegative matrix as the first one fails to be reducible as there is a repeating Perron root and the second one fails the moment condition as $s_3 \ngeqslant 0$. Therefore we denote by $S_n$ the set of points in $\tau_n$ that are realized as spectrums of symmetric, nonnegative matrices.

Soules [31] developed a method of constructing symmetric nonnegative matrices whose eigenvalues satisfy $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$. Starting with a vector $x > 0$, Soules shows how to construct an orthogonal matrix $R$, called the Soules matrix, such that for $\Delta = diag\,(\lambda_1, \lambda_2, \ldots, \lambda_n)$ the matrix $A = R\Delta R^T$ has nonnegative off-diagonal entries. He shows that all off-diagonal entries are positive if and only if $\lambda_1 > \lambda_2$. Soules then finds sufficient conditions on $\lambda_i$'s which guarantee the diagonal entries of $A = R\Delta R^T$ are also nonnegative. In this construction from Chapter 5 Theorem 23, the positive vector $x$ becomes the Perron vector of $A$ corresponding to $\lambda_1$.

Let the Soules set $\mathfrak{S}_n$ be the set of all lists $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ for which there is a

Soules matrix $R$ such that $R\Delta R^T \geqslant 0$ where $\Delta = diag\,(\lambda_1, \lambda_2, \ldots, \lambda_n)$. For $n = 3$ and $n = 4$, McDonald and Newmann [21] show that $\mathfrak{S}_n = S_n = \tau_n$. In Knudsen and McDonald it is shown that $\{1, \frac{1+3\sqrt{5}}{16}, \frac{1+3\sqrt{5}}{16}, \frac{-7-3\sqrt{5}}{16}, \frac{-7-3\sqrt{5}}{16}\}$ is in $S_5$ but not in $\mathfrak{S}_5$ showing that $\mathfrak{S}_5$ is properly contained in $S_5$. We saw earlier that $S_5$ is properly contained in $\tau_5$.

The following interesting lemma by McDonald and Neumann [21] gives a condition relative to the spectrum of a 5 x 5 irreducible symmetric nonnegative matrix. Although it is a simple condition, it is a useful result for our work in section (6.3).

**Lemma 1.** [21] *Let $A$ be a 5 x 5 irreducible nonnegative symmetric matrix with eigenvalues $\{1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\}$ where $1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \lambda_4 \geqslant \lambda_5 \geqslant -1$. Then $-(\lambda_5) + trace\,(A) \geqslant \lambda_2$. Moreover, if some eigenvector of $A$ corresponding to $\lambda_2$ has only one positive or one negative entry, then $trace\,(A) \geqslant \lambda_2$.*

*Proof.* If $\lambda_5 \geqslant 0$, then $-\lambda_5 + trace\,(A) = 1 + \lambda_2 + \lambda_3 + \lambda_4 \geqslant \lambda_2$ since $A$ has all nonnegative eigenvalues.

If $\lambda_5 < 0$, then let $B = A - \lambda_5 I$ where $A = [a_{i,j}]$ for $i, j = 1, 2, 3, 4, 5$. By this definition, since $B$ has all nonnegative eigenvalues and is Hermitian, $B$ is a positive semidefinite matrix. Since $\lambda_5$ is negative, $B$ is also a symmetric nonnegative matrix. Let $x$ be the Perron vector for the Perron root, $1 - \lambda_5$, associated with matrix $B$. Then by the Perron-Frobenius Theorem $x > 0$ and can be chosen so that $\|x\|_2 = 1$. Let $y$ be the eigenvector corresponding to $\lambda_2 - \lambda_5$ associated with matrix $B$. Since $B$ is symmetric, $x^T y = 0$. This forces $y$ to have at least one negative entry and at least

one positive entry. As $y$ has five entries, this means $y$ or $-y$ has at most two positive entries. Let us assume $y$ has at most two positive entries located at the top of the vector. Now let $y$ be represented by the two vectors $y_1$ and $y_2$ where $y_1$ contains the positive part of $y$ and $y_2$ contains the nonpositive part of $y$. Partition $B$ appropriately for $y$ so that

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}.$$

Consider the two cases:

1. Suppose $y_1$ contains exactly two elements. Then we partition $B$ so that $B_{11}$ is 2 x 2 and $B_{22}$ is 3 x 3. Thus we have $B_{11}y_1 + B_{12}y_2 = (\lambda_2 - \lambda_5)\, y_1$. Then as $B_{12}y_2 \leqslant 0$, we have $B_{11}y_1 \geqslant (\lambda_2 - \lambda_5)\, y_1$. Since $B$ is nonnegative and semidefinite, so is $B_{11}$ and we can conclude that by the definition of $B = A - \lambda_5 I$ that

$$\begin{aligned} trace\,(A) - 2\lambda_5 &\geqslant a_{11} + a_{22} - 2\lambda_5 \\ &= trace\,(B_{11}) \\ &\geqslant \rho\,(B_{11}) \\ &\geqslant \lambda_2 - \lambda_5. \end{aligned}$$

Therefore, $-(\lambda_5) + trace\,(A) \geqslant \lambda_2$.

2. Suppose $y_1$ contains one element. Then by a similar argument where $B_{11}$ is a 1 x 1 matrix, we have

$$\begin{aligned} trace\,(A) - \lambda_5 &\geqslant a_{11} - \lambda_5 \\ &= trace\,(B_{11}) \\ &\geqslant \rho\,(B_{11}) \\ &\geqslant \lambda_2 - \lambda_5 \end{aligned}$$

giving $trace\,(A) \geqslant \lambda_2$.

$\square$

In section (6.3) we show that, except for two partially known cases, every other possible list of five real numbers satisfying certain sufficient conditions are in $S_5$.

## 6.2 Equivalence of RNIEP and SNIEP for $n \leqslant 4$

In the case of lists of four real numbers, Loewy and London [20] have a construction technique solving the RNIEP. This is discussed in Theorem 28. The following symmetric constructions are from Wuwen's paper [34]. In these constructions, Wuwen uses Theorem 27 due to Fiedler.

1. Consider a list of length one. Clearly $\lambda_1 \geqslant 0$ and hence $A = \begin{bmatrix} \lambda_1 \end{bmatrix}$ will do.
2. For a list of length two, $\sigma = \{\lambda_1,\ \lambda_2\}$ must satisfy $\lambda_1 + \lambda_2 \geqslant 0$ where $\lambda_1 \geqslant |\lambda_2|$.

   An example of a symmetric nonnegative matrix with $\sigma$ as its spectrum is:

$$A = \frac{1}{2}\begin{bmatrix} \lambda_1 + \lambda_2 & \lambda_1 - \lambda_2 \\ \lambda_1 - \lambda_2 & \lambda_1 + \lambda_2 \end{bmatrix}$$

3. Now we consider a list of length three. In this case where $\sigma = \{\lambda_1,\ \lambda_2,\ \lambda_3\} \subset R$, $\lambda_1 + \lambda_2 + \lambda_3 \geqslant 0$, $\lambda_1 \geqslant |\lambda_i|$ for $i = 2,\ 3$ and $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3$ the following possibilities arise:

   (a) if $\lambda_1 \geqslant \lambda_2 \geqslant 0 \geqslant \lambda_3$, then an example of a symmetric nonnegative matrix realizing $\sigma$ as its spectrum is:

$$A = \begin{bmatrix} \frac{\lambda_1+\lambda_3}{2} & \frac{\lambda_1-\lambda_3}{2} & 0 \\ \frac{\lambda_1-\lambda_3}{2} & \frac{\lambda_1+\lambda_3}{2} & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix}$$

   (b) if $\lambda_1 \geqslant 0 \geqslant \lambda_2 \geqslant \lambda_3$, then an example of a symmetric nonnegative matrix realizing $\sigma$ as its spectrum is:

$$A = \begin{bmatrix} \frac{\lambda_1+\lambda_2+\lambda_3}{2} & \frac{\lambda_1+\lambda_2-\lambda_3}{2} & \frac{\sqrt{-2\lambda_1\lambda_2}}{2} \\ \frac{\lambda_1+\lambda_2-\lambda_3}{2} & \frac{\lambda_1+\lambda_2+\lambda_3}{2} & \frac{\sqrt{-2\lambda_1\lambda_2}}{2} \\ \frac{\sqrt{-2\lambda_1\lambda_2}}{2} & \frac{\sqrt{-2\lambda_1\lambda_2}}{2} & 0 \end{bmatrix}.$$

4. Next we present two results for the case of $n = 4$. Here $\sigma = \{\lambda_1,\ \lambda_2,\ \lambda_3,\ \lambda_4\} \subset R$, $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \lambda_4$, and $\lambda_1 \geqslant |\lambda_i|$ for $i = 2,\ 3,\ 4$. The RNIEP solution given in the next theorem is due to Loewy and London [20].

**Theorem 28.** [20] *Let $\sigma = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ be a set of four real numbers, and assume that $\sigma$ satisfies $max_{1\leqslant i\leqslant 4} |\lambda_i| \in \sigma$ and $s_1 = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 \geqslant 0$. Then $\sigma$ is the set of eigenvalues of a nonnegative matrix $A$.*

*Proof.* Assume that $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \lambda_4$ and we have the Perron root requirement, $|\lambda_i| \leqslant \lambda_1$, for $i = 2, 3, 4$.

If $\lambda_2, \lambda_3, \lambda_4 > 0$, $A$ is the matrix $A = diag\,(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$.

If $\lambda_2, \lambda_3, \lambda_4 \leqslant 0$, the assertion follows from Theorem 16.

If $\lambda_4 \leqslant 0$, $\lambda_2, \lambda_3 > 0$, $\sigma$ is realized by the nonnegative matrix

$$A = \begin{bmatrix} \lambda_1 + \lambda_4 & \lambda_1 & 0 & 0 \\ -\lambda_4 & 0 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{bmatrix}$$

If $\lambda_3, \lambda_4 \leqslant 0$, $\lambda_2 > 0$ we distinguish between two cases:

(a) $\lambda_2 + \lambda_3 \geqslant 0$, and

(b) $\lambda_2 + \lambda_3 < 0$.

In case (a), $\sigma$ is realized by the nonnegative matrix

$$A = \begin{bmatrix} \lambda_2 + \lambda_3 & \lambda_3 & 0 & 0 \\ -\lambda_3 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1 + \lambda_4 & \lambda_1 \\ 0 & 0 & -\lambda_4 & 0 \end{bmatrix}$$

In case (b), $\sigma$ is realized by the matrix $A = Udiag(\lambda_1, \lambda_2, \lambda_3, \lambda_4)\, U^T$ where $U$ is the orthogonal matrix

$$U = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 \end{bmatrix}.$$

The entries of $A$ are,

$$a_{11} = a_{22} = a_{33} = a_{44} = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 \geqslant 0,$$

$$a_{12} = a_{21} = a_{34} = a_{43} = -\lambda_1 - \lambda_2 + \lambda_3 + \lambda_4 \geqslant 0,$$

$$a_{13} = a_{31} = a_{24} = a_{42} = \lambda_1 - \lambda_2 - \lambda_3 + \lambda_4 \geqslant 0,$$

$$a_{14} = a_{41} = a_{23} = a_{32} = -\lambda_1 + \lambda_2 - \lambda_3 + \lambda_4 \geqslant 0,$$

and hence $A$ is a nonnegative matrix. □

The solution to the SNIEP for a list of four real numbers was presented by Wuwen [34]. The construction techniques require the same sufficient conditions as those in Theorem 28.

(a) If $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \lambda_4 \geqslant 0$ then $A = diag(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ realizes $\sigma$ as its spectrum.

(b) If $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant 0 > \lambda_4$, then an example of a symmetric nonnegative

matrix realizing $\sigma$ as its spectrum is:

$$A = \frac{1}{2}\begin{bmatrix} \lambda_1 + \lambda_4 & \lambda_1 - \lambda_4 & 0 & 0 \\ \lambda_1 - \lambda_4 & \lambda_1 + \lambda_4 & 0 & 0 \\ 0 & 0 & 2\lambda_2 & 0 \\ 0 & 0 & 0 & 2\lambda_3 \end{bmatrix}.$$

(c) If $\lambda_1 \geqslant \lambda_2 \geqslant 0 > \lambda_3 \geqslant \lambda_4$ and $\lambda_2 \geqslant |\lambda_3|$, then an example of a symmetric nonnegative matrix realizing $\sigma$ as its spectrum is:

$$A = \frac{1}{2}\begin{bmatrix} \lambda_1 + \lambda_4 & \lambda_1 - \lambda_4 & 0 & 0 \\ \lambda_1 - \lambda_4 & \lambda_1 + \lambda_4 & 0 & 0 \\ 0 & 0 & \lambda_2 + \lambda_3 & \lambda_2 - \lambda_3 \\ 0 & 0 & \lambda_2 - \lambda_3 & \lambda_2 + \lambda_3 \end{bmatrix}.$$

(d) If $\lambda_1 \geqslant \lambda_2 \geqslant 0 > \lambda_3 \geqslant \lambda_4$ and $\lambda_2 < |\lambda_3|$, then an example of a symmetric nonnegative matrix realizing $\sigma$ as its spectrum is:

$$A = \frac{1}{2}\begin{bmatrix} \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 & \lambda_1 + \lambda_2 + \lambda_3 - \lambda_4 & \delta & \delta \\ \lambda_1 + \lambda_2 + \lambda_3 - \lambda_4 & \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 & \delta & \delta \\ \delta & \delta & 0 & -2\lambda_3 \\ \delta & \delta & -2\lambda_3 & 0 \end{bmatrix},$$

where $\delta = \sqrt{-(\lambda_2 + \lambda_3)(\lambda_1 + \lambda_3)}$.

(e) If $\lambda_1 \geqslant 0 > \lambda_2 \geqslant \lambda_3 \geqslant \lambda_4$, using

$$B = \frac{1}{2}\begin{bmatrix} \lambda_1+\lambda_2+\lambda_3+\lambda_4 & \lambda_1+\lambda_2+\lambda_3+\lambda_4 & \sqrt{-2\left(\lambda_1+\lambda_2\right)\lambda_3} \\ \lambda_1+\lambda_2+\lambda_3+\lambda_4 & \lambda_1+\lambda_2+\lambda_3+\lambda_4 & \sqrt{-2\left(\lambda_1+\lambda_2\right)\lambda_3} \\ \sqrt{-2\left(\lambda_1+\lambda_2\right)\lambda_3} & \sqrt{-2\left(\lambda_1+\lambda_2\right)\lambda_3} & 0 \end{bmatrix},$$

$$u = \frac{1}{\sqrt{2\left(\lambda_1+\lambda_2-\lambda_3\right)}}\begin{bmatrix} \sqrt{\lambda_1+\lambda_2} \\ \sqrt{\lambda_1+\lambda_2} \\ \sqrt{-2\lambda_3} \end{bmatrix},$$

and $\delta = \sqrt{-\lambda_1\lambda_2}$, then an example of a symmetric nonnegative matrix realizing $\sigma$ as its spectrum is:

$$A = \begin{bmatrix} B & \delta u \\ \delta u^T & 0 \end{bmatrix}.$$

This completes the case where the RNIEP and the SNIEP are equivalent. Although Johnson, Laffey and Loewy [13] proved the two problems are not equivalent for all dimensions, it was an existence proof as described in Chapter 5. In section (6.3) we analyze the case where $\sigma = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\}$ in terms of the RNIEP and the SNIEP. We show that the list of five real numbers is the first time the two problems are not equivalent.

## 6.3 Lists of Five Real Numbers

The following addresses the SNIEP in the case where the lists contain five real numbers. That is, given a list of five real numbers $\sigma = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\}$ such that

1. $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \lambda_4 \geqslant \lambda_5$,

2. $\lambda_1 = max_{1 \leqslant i \leqslant 5} |\lambda_i|$,

3. $s_k \geqslant 0$ for $k = 1, 2, 3, \ldots$ and

4. JLL condition

we determine sufficient conditions for $\sigma$ to be the spectrum of a 5 x 5 symmetric nonnegative matrix. In the analysis, we first attempt to use the conditions from Kellogg's Theorem to determine if there exists a nonnegative matrix and thus a symmetric nonnegative matrix due to a result of Fiedler. We do so by examining the pairing of positive numbers in the list with nonpositive numbers in the list. For example if $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 > 0 \geqslant \lambda_4 \geqslant \lambda_5$ then the pairs in question are $\lambda_2$ with $\lambda_5$ and $\lambda_3$ with $\lambda_4$. If $\lambda_2 + \lambda_5 < 0$ and $\lambda_3 + \lambda_4 < 0$ then the set $K = \{2, 3\}$ in Kellogg's conditions. The remaining conditions are then satisfied if $\lambda_1 + \lambda_5 \geqslant 0$, $\lambda_1 + (\lambda_2 + \lambda_5) + \lambda_4 \geqslant 0$, and $\lambda_1 + (\lambda_2 + \lambda_5) + (\lambda_3 + \lambda_4) \geqslant 0$. If there are cases where Kellogg's conditions do not apply, for instance in our example when $\lambda_1 + (\lambda_2 + \lambda_5) + \lambda_4 < 0$, then we attempt to partition $\sigma$ and use the conditions from Borobia's Theorem or Radwan's Theorem to see if there exists a symmetric nonnegative matrix. The final goal of each case is a construction technique showing a symmetric nonnegative matrix realizing $\sigma$ as its spectrum. Many times we will use Fiedler's result from Theorem 27.

The following figures show the possible positive and nonpositive distribution of five real numbers. The break down from each are the possible outcomes based on

Kellogg's conditions. Each branch further divides out when we apply Borobia's partitioning scheme.

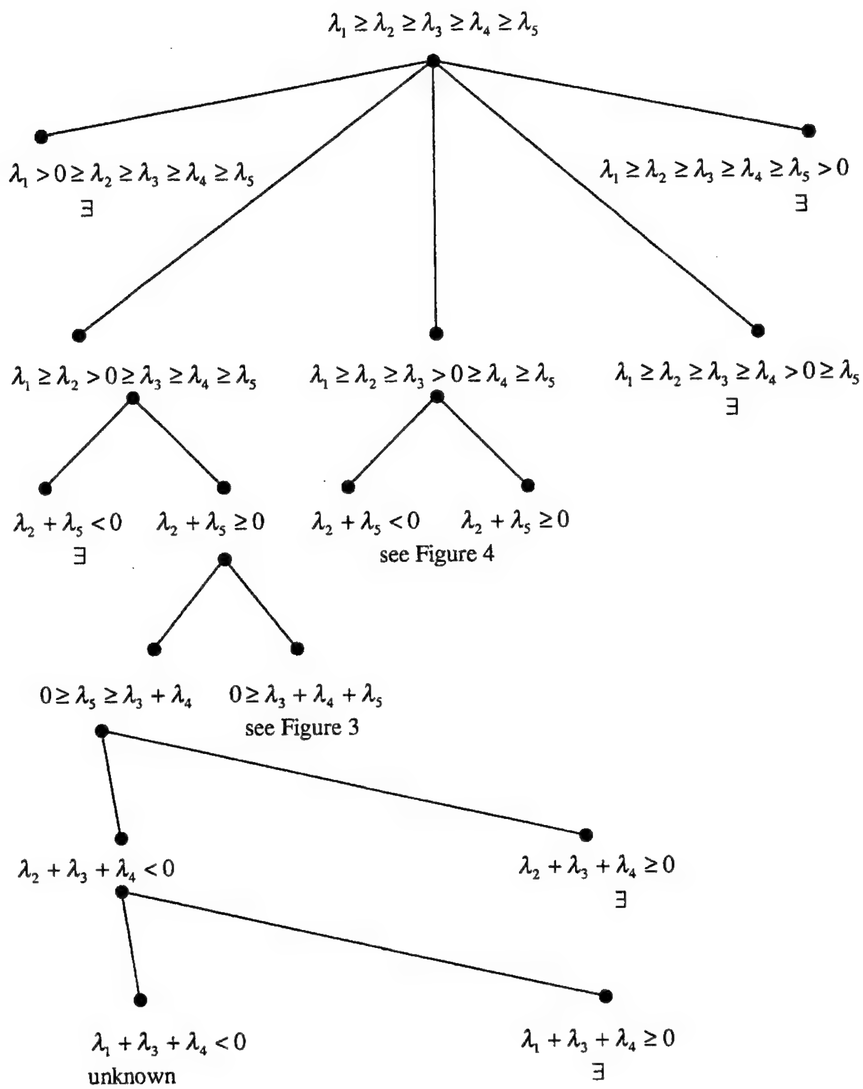


Figure 2: List of Five Real Numbers

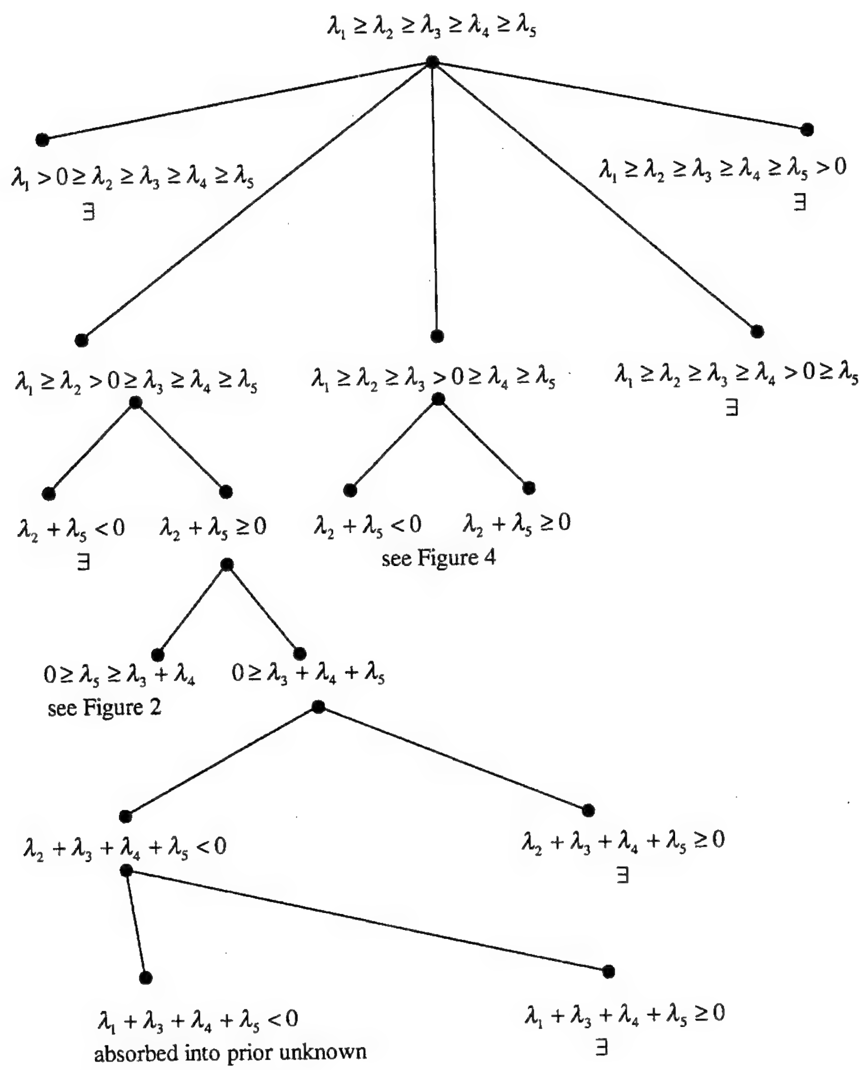


Figure 3: List of Five Real Numbers

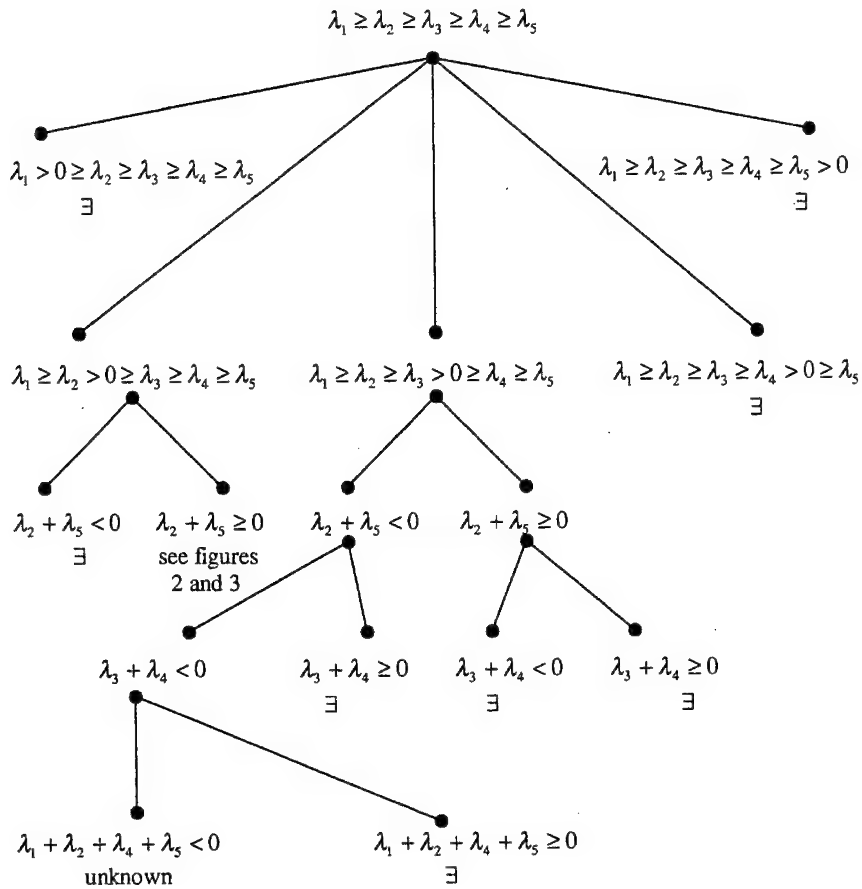


Figure 4: List of Five Real Numbers

Our answers to the SNIEP for lists of five real numbers lie in the results of the following cases.

### 6.3.1 $\lambda_1 > 0 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \lambda_4 \geqslant \lambda_5$

By Suleimanova's Theorem and Theorem 27, there exists a 5 x 5 nonnegative, symmetric matrix having spectrum $\{\lambda_1,\ \lambda_2,\ \lambda_3,\ \lambda_4,\ \lambda_5\}$. A construction is as follows:

$$A_1 = \begin{bmatrix} \frac{1}{2}(\lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5) & \frac{1}{2}(\lambda_1+\lambda_2+\lambda_3+\lambda_4-\lambda_5) & \frac{1}{2}\sqrt{-2(\lambda_1+\lambda_2+\lambda_3)\lambda_4} & a \\ \frac{1}{2}(\lambda_1+\lambda_2+\lambda_3+\lambda_4-\lambda_5) & \frac{1}{2}(\lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5) & \frac{1}{2}\sqrt{-2(\lambda_1+\lambda_2+\lambda_3)\lambda_4} & a \\ \frac{1}{2}\sqrt{-2(\lambda_1+\lambda_2+\lambda_3)\lambda_4} & \frac{1}{2}\sqrt{-2(\lambda_1+\lambda_2+\lambda_3)\lambda_4} & 0 & b \\ a & a & b & 0 \end{bmatrix}$$

where $a = \frac{\sqrt{-(\lambda_1+\lambda_2)\lambda_3}\sqrt{\lambda_1+\lambda_2+\lambda_3}}{\sqrt{2(\lambda_1+\lambda_2+\lambda_3-\lambda_4)}}$, $b = \frac{\sqrt{-(\lambda_1+\lambda_2)\lambda_3}\sqrt{-2\lambda_4}}{\sqrt{2(\lambda_1+\lambda_2+\lambda_3-\lambda_4)}}$. Since $A_1$ has eigenvalues $\lambda_1+\lambda_2$, $\lambda_3$, $\lambda_4$, and $\lambda_5$ where $\lambda_1+\lambda_2$ is the Perron root, we know that the normalized Perron vector $u$ of $\lambda_1+\lambda_2$ is nonnegative. So using Theorem 27, with matrix $B=[0]$, which has Perron root 0 and Perron vector $v=[1]$, we find $\rho=\sqrt{-\lambda_1\lambda_2} \geqslant 0$ which makes the following matrix symmetric and nonnegative. Thus,

$$A = \begin{bmatrix} A_1 & \rho u v^T \\ \rho v u^T & 0 \end{bmatrix}$$

is nonnegative and has the spectrum $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\}$.

### 6.3.2 $\lambda_1 \geqslant \lambda_2 > 0 \geqslant \lambda_3 \geqslant \lambda_4 \geqslant \lambda_5$

1. If $\lambda_2+\lambda_5<0$ then Kellogg's conditions are satisfied. That is $\lambda_1+\lambda_5 \geqslant 0$ and $\lambda_1+(\lambda_2+\lambda_5)+\lambda_3+\lambda_4 \geqslant 0$ because the first is the Perron root condition and the second is the trace nonnegative condition. This means $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\}$ is the spectrum of some nonnegative symmetric matrix. An example of such a matrix using Theorem 27 is

$$A = \frac{1}{2}\begin{bmatrix} \lambda_1-\varepsilon+\lambda_3+\lambda_4 & \lambda_1-\varepsilon+\lambda_3-\lambda_4 & \sqrt{-2(\lambda_1-\varepsilon)\lambda_3} & \delta & \delta \\ \lambda_1-\varepsilon+\lambda_3-\lambda_4 & \lambda_1-\varepsilon+\lambda_3+\lambda_4 & \sqrt{-2(\lambda_1-\varepsilon)\lambda_3} & \delta & \delta \\ \sqrt{-2(\lambda_1-\varepsilon)\lambda_3} & \sqrt{-2(\lambda_1-\varepsilon)\lambda_3} & 0 & \rho & \rho \\ \delta & \delta & \rho & \lambda_2+\varepsilon+\lambda_5 & \lambda_2+\varepsilon-\lambda_5 \\ \delta & \delta & \rho & \lambda_2+\varepsilon-\lambda_5 & \lambda_2+\varepsilon+\lambda_5 \end{bmatrix}$$

   where $\delta = \frac{\sqrt{\varepsilon}(\lambda_1-\varepsilon)\sqrt{\lambda_1-\varepsilon-\lambda_2}}{2\sqrt{(\varepsilon-\lambda_1)\lambda_3}\sqrt{\frac{\varepsilon-\lambda_1+\lambda_3}{\lambda_3}}}$, $\rho = \frac{\sqrt{\varepsilon}\sqrt{\lambda_1-\varepsilon-\lambda_2}}{\sqrt{2}\sqrt{\frac{\varepsilon-\lambda_1+\lambda_3}{\lambda_3}}}$, and $\varepsilon \geqslant 0$ satisfying $\lambda_1-\varepsilon \geqslant \lambda_2$ and $\lambda_2+\varepsilon+\lambda_5 \geqslant 0$.

2. For all remaining subcases, $\lambda_2+\lambda_5 \geqslant 0$ must be true. So one of the following partitions of the set $\sigma$ must apply.

(a) If $\lambda_1 \geqslant \lambda_2 > 0 \geqslant \lambda_5 \geqslant \lambda_3 + \lambda_4$ and $\lambda_2 + \lambda_3 + \lambda_4 \geqslant 0$. Then the following matrix is symmetric, nonnegative, and has spectrum $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\}$.

$$A = \frac{1}{2}\begin{bmatrix} \lambda_2+\lambda_3+\lambda_4 & \lambda_2+\lambda_3-\lambda_4 & \sqrt{-2\lambda_2\lambda_3} & 0 & 0 \\ \lambda_2+\lambda_3-\lambda_4 & \lambda_2+\lambda_3+\lambda_4 & \sqrt{-2\lambda_2\lambda_3} & 0 & 0 \\ \sqrt{-2\lambda_2\lambda_3} & \sqrt{-2\lambda_2\lambda_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1+\lambda_5 & \lambda_1-\lambda_5 \\ 0 & 0 & 0 & \lambda_1-\lambda_5 & \lambda_1+\lambda_5 \end{bmatrix}$$

(b) If $\lambda_1 \geqslant \lambda_2 > 0 \geqslant \lambda_5 \geqslant \lambda_3+\lambda_4$ where $\lambda_2+\lambda_3+\lambda_4 < 0$ but $\lambda_1+\lambda_3+\lambda_4 \geqslant 0$, then the following matrix is symmetric, nonnegative and has the desired spectrum $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\}$.

$$A = \frac{1}{2}\begin{bmatrix} \lambda_2+\lambda_5 & \lambda_2-\lambda_5 & 0 & 0 & 0 \\ \lambda_2-\lambda_5 & \lambda_2+\lambda_5 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1+\lambda_3+\lambda_4 & \lambda_1+\lambda_3-\lambda_4 & \sqrt{-2(\lambda_1+\lambda_3)\lambda_4} \\ 0 & 0 & \lambda_1+\lambda_3-\lambda_4 & \lambda_1+\lambda_3+\lambda_4 & \sqrt{-2(\lambda_1+\lambda_3)\lambda_4} \\ 0 & 0 & \sqrt{-2(\lambda_1+\lambda_3)\lambda_4} & \sqrt{-2(\lambda_1+\lambda_3)\lambda_4} & 0 \end{bmatrix}$$

(c) If $\lambda_1 \geqslant \lambda_2 > 0 \geqslant \lambda_5 \geqslant \lambda_3 + \lambda_4$ and no previous conditions such as (2a) or (2b), then we can form certain conclusions about the problem. First, if a solution exists, it must be an irreducible matrix. Second, $\lambda_3 \neq 0$. Third the two restrictions that $0 > \lambda_5 > \lambda_3 + \lambda_4$ and $\lambda_2 + \lambda_5 > 0$ must apply.

For the first conclusion, we rule out a repeating Perron root. Suppose $\lambda_1 = \lambda_2$, then the partitioning $\{\lambda_1, \lambda_5\}$ and $\{\lambda_2, \lambda_3, \lambda_4\}$, must be spectra of 2 x 2 and 3 x 3 nonnegative matrices respectively. But then we have $\lambda_1 + \lambda_3 + \lambda_4 = \lambda_2 + \lambda_3 + \lambda_4 < 0$ which contradicts the necessary trace condition. In the case of $\lambda_1 \neq \lambda_2$, if the matrix is reducible some partition

must occur among the list of five numbers. However the weakest partition of $\{\lambda_1, \lambda_3, \lambda_4\}$ cannot occur since we are not assuming (2b). But any other partition is worse since $\lambda_3 + \lambda_4 \geqslant \lambda_i + \lambda_j$ for any $i, j = 3, 4, 5$ with $i \neq j$. This means $\lambda_1$ will never dominate the sum of any two of the nonpositive entries; therefore, no other partition will work.

The second conclusion that $\lambda_3 \neq 0$ is because $\lambda_1 + \lambda_3 + \lambda_4 < 0$. As we must satisfy the Perron root condition that $\lambda_1 + \lambda_4 \geqslant 0$, adding a zero will not make $\lambda_1 + \lambda_3 + \lambda_4 < 0$.

In the third conclusion, the first restriction is derived from the inequalities $0 > \lambda_3 \geqslant \lambda_4 \geqslant \lambda_5$ and $\lambda_1 + \lambda_3 + \lambda_4 < 0$. As we no longer have Perron root domination with this partition and $\lambda_1 + \lambda_5 \geqslant 0$, we realize that $\lambda_5 > \lambda_3 + \lambda_4$. The second restriction is apparent from the nonnegative trace condition, $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 \geqslant 0$. For if $\lambda_2 + \lambda_5 = 0$ then $\lambda_1 + \lambda_3 + \lambda_4 \geqslant 0$ contradicting $\lambda_1 + \lambda_3 + \lambda_4 < 0$. Hence $\lambda_2 + \lambda_5 > 0$.

An additional observation is that this case requires $2\lambda_2 > \lambda_1$. We find this to be true since $\lambda_2 + \lambda_3 > 0$ and $\lambda_2 + \lambda_4 > 0$ giving $2\lambda_2 > -(\lambda_3 + \lambda_4)$ and since $\lambda_1 + \lambda_3 + \lambda_4 < 0$ giving $-(\lambda_3 + \lambda_4) > \lambda_1$. Putting it all together we have $2\lambda_2 > -(\lambda_3 + \lambda_4) > \lambda_1$.

Therefore, the complete set of restrictions for this case are:

i. $\lambda_1 > \lambda_2 > 0 > \lambda_3 \geqslant \lambda_4 \geqslant \lambda_5$

ii. $\lambda_2 + \lambda_5 > 0$

iii. $\lambda_3 + \lambda_4 < \lambda_5$, and

iv. $\lambda_1 + \lambda_3 + \lambda_4 < 0$.

As of this submission, we have examples of lists satisfying the prescribed restrictions and realized by nonnegative matrices and also examples of lists that meet the restrictions that cannot be spectrums of any nonnegative matrix. We prove that there does not exist any symmetric nonnegative matrix having spectrum with the prescribed restrictions and also prove that if a nonnegative matrix does exist, it must be irreducible. The proof and details are in the conclusion section (6.4) of this chapter.

(d) If $\lambda_1 \geqslant \lambda_2 > 0 \geqslant \lambda_3+\lambda_4+\lambda_5$ and $\lambda_2+\lambda_3+\lambda_4+\lambda_5 \geqslant 0$, then the following matrix is symmetric, nonnegative, and has spectrum $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\}$.

$$A = \frac{1}{2}\begin{bmatrix} \lambda_2+\lambda_3+\lambda_4+\lambda_5 & \lambda_2+\lambda_3+\lambda_4-\lambda_5 & \sqrt{-2(\lambda_2+\lambda_3)\lambda_4} & \frac{\sqrt{-\lambda_2\lambda_3}\sqrt{2(\lambda_2+\lambda_3)}}{\sqrt{\lambda_2+\lambda_3-\lambda_4}} & 0 \\ \lambda_2+\lambda_3+\lambda_4-\lambda_5 & \lambda_2+\lambda_3+\lambda_4+\lambda_5 & \sqrt{-2(\lambda_2+\lambda_3)\lambda_4} & \frac{\sqrt{-\lambda_2\lambda_3}\sqrt{2(\lambda_2+\lambda_3)}}{\sqrt{\lambda_2+\lambda_3-\lambda_4}} & 0 \\ \sqrt{-2(\lambda_2+\lambda_3)\lambda_4} & \sqrt{-2(\lambda_2+\lambda_3)\lambda_4} & 0 & \frac{2\sqrt{\lambda_2\lambda_3\lambda_4}}{\sqrt{\lambda_2+\lambda_3-\lambda_4}} & 0 \\ \frac{\sqrt{-\lambda_2\lambda_3}\sqrt{2(\lambda_2+\lambda_3)}}{\sqrt{\lambda_2+\lambda_3-\lambda_4}} & \frac{\sqrt{-\lambda_2\lambda_3}\sqrt{2(\lambda_2+\lambda_3)}}{\sqrt{\lambda_2+\lambda_3-\lambda_4}} & \frac{2\sqrt{\lambda_2\lambda_3\lambda_4}}{\sqrt{\lambda_2+\lambda_3-\lambda_4}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\lambda_1 \end{bmatrix}$$

(e) If $\lambda_1 \geqslant \lambda_2 > 0 \geqslant \lambda_3+\lambda_4+\lambda_5$, $\lambda_2+\lambda_3+\lambda_4+\lambda_5 < 0$, and $\lambda_1+\lambda_3+\lambda_4+\lambda_5 \geqslant 0$, then the following symmetric nonnegative matrix has spectrum $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\}$.

$$A = \frac{1}{2}\begin{bmatrix} \lambda_1+\lambda_3+\lambda_4+\lambda_5 & \lambda_1+\lambda_3+\lambda_4-\lambda_5 & \sqrt{-2(\lambda_1+\lambda_3)\lambda_4} & \frac{\sqrt{-\lambda_1\lambda_3}\sqrt{2(\lambda_1+\lambda_3)}}{\sqrt{\lambda_1+\lambda_3-\lambda_4}} & 0 \\ \lambda_1+\lambda_3+\lambda_4-\lambda_5 & \lambda_1+\lambda_3+\lambda_4+\lambda_5 & \sqrt{-2(\lambda_1+\lambda_3)\lambda_4} & \frac{\sqrt{-\lambda_1\lambda_3}\sqrt{2(\lambda_1+\lambda_3)}}{\sqrt{\lambda_1+\lambda_3-\lambda_4}} & 0 \\ \sqrt{-2(\lambda_1+\lambda_3)\lambda_4} & \sqrt{-2(\lambda_1+\lambda_3)\lambda_4} & 0 & \frac{2\sqrt{\lambda_1\lambda_3\lambda_4}}{\sqrt{\lambda_1+\lambda_3-\lambda_4}} & 0 \\ \frac{\sqrt{-\lambda_1\lambda_3}\sqrt{2(\lambda_1+\lambda_3)}}{\sqrt{\lambda_1+\lambda_3-\lambda_4}} & \frac{\sqrt{-\lambda_1\lambda_3}\sqrt{2(\lambda_1+\lambda_3)}}{\sqrt{\lambda_1+\lambda_3-\lambda_4}} & \frac{2\sqrt{\lambda_1\lambda_3\lambda_4}}{\sqrt{\lambda_1+\lambda_3-\lambda_4}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\lambda_2 \end{bmatrix}$$

### 6.3.3 $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 > 0 \geqslant \lambda_4 \geqslant \lambda_5$

(a) If $\lambda_2 + \lambda_5 < 0$, $\lambda_3 + \lambda_4 < 0$, and $\lambda_1 + \lambda_2 + \lambda_4 + \lambda_5 \geqslant 0$, then Kellogg's conditions are satisfied and by Theorem 27, there exists a symmetric non-negative matrix having spectrum $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\}$. An example of such is the following matrix.

$$A = \frac{1}{2}\begin{bmatrix} \lambda_1+\lambda_3+\lambda_4+\lambda_5 & \lambda_1+\lambda_2+\lambda_4-\lambda_5 & \delta & \delta & 0 \\ \lambda_1+\lambda_2+\lambda_4-\lambda_5 & \lambda_1+\lambda_2+\lambda_4+\lambda_5 & \delta & \delta & 0 \\ \delta & \delta & 0 & -2\lambda_4 & 0 \\ \delta & \delta & -2\lambda_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\lambda_3 \end{bmatrix}$$

where $\delta = \sqrt{-(\lambda_2 + \lambda_4)(\lambda_1 + \lambda_4)}$.

(b) If $\lambda_2 + \lambda_5 < 0$, $\lambda_3 + \lambda_4 \geqslant 0$, then the following matrix has spectrum $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\}$.

$$A = \frac{1}{2}\begin{bmatrix} \lambda_1+\lambda_5 & \lambda_1-\lambda_5 & 0 & 0 & 0 \\ \lambda_1-\lambda_5 & \lambda_1+\lambda_5 & 0 & 0 & 0 \\ 0 & 0 & \lambda_3+\lambda_4 & \lambda_3-\lambda_4 & 0 \\ 0 & 0 & \lambda_3-\lambda_4 & \lambda_3+\lambda_4 & 0 \\ 0 & 0 & 0 & 0 & 2\lambda_2 \end{bmatrix}$$

(c) If $\lambda_2 + \lambda_5 \geqslant 0$, $\lambda_3 + \lambda_4 < 0$, then the following matrix has spectrum

$\{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\}$.

$$A = \frac{1}{2}\begin{bmatrix} \lambda_2+\lambda_5 & \lambda_2-\lambda_5 & 0 & 0 & 0 \\ \lambda_2-\lambda_5 & \lambda_2+\lambda_5 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1+\lambda_4 & \lambda_1-\lambda_4 & 0 \\ 0 & 0 & \lambda_1-\lambda_4 & \lambda_1+\lambda_4 & 0 \\ 0 & 0 & 0 & 0 & 2\lambda_3 \end{bmatrix}$$

(d) If $\lambda_2 + \lambda_5 \geqslant 0$, $\lambda_3 + \lambda_4 \geqslant 0$, then the following matrix has spectrum $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\}$.

$$A = \frac{1}{2}\begin{bmatrix} \lambda_2+\lambda_5 & \lambda_2-\lambda_5 & 0 & 0 & 0 \\ \lambda_2-\lambda_5 & \lambda_2+\lambda_5 & 0 & 0 & 0 \\ 0 & 0 & \lambda_3+\lambda_4 & \lambda_3-\lambda_4 & 0 \\ 0 & 0 & \lambda_3-\lambda_4 & \lambda_3+\lambda_4 & 0 \\ 0 & 0 & 0 & 0 & 2\lambda_1 \end{bmatrix}$$

(e) If $\lambda_2 + \lambda_5 < 0$, $\lambda_3 + \lambda_4 < 0$, and $\lambda_1 + \lambda_2 + \lambda_4 + \lambda_5 < 0$ where $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 > 0 \geqslant \lambda_4 \geqslant \lambda_5$, then Kellogg's conditions are not satisfied. For Borobia's conditions partitioning can only happen in one way; namely, $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 > 0 > \lambda_4 + \lambda_5$.

Moreover, we can deduce additional restrictions. For if $\lambda_1 = \lambda_2$ it is not possible to partition the list to get a reducible matrix realizing the lists as its spectrum. If $\lambda_4 = 0$ then $\lambda_1+\lambda_2+\lambda_4+\lambda_5 < 0$ becomes $\lambda_1+\lambda_2+\lambda_5 < 0$. Since $\lambda_1 + \lambda_5 \geqslant 0$ and $\lambda_2 > 0$ this cannot happen. Therefore we have the

following inequalities $\lambda_1 > \lambda_2 \geqslant \lambda_3 > 0 > \lambda_4 \geqslant \lambda_5$.

Borobia's conditions are not satisfied in this partition where we group $\lambda_4$ and $\lambda_5$ together because the Perron root does not dominate the sum of $\lambda_4$ and $\lambda_5$. This is apparent from the restriction $\lambda_1 + \lambda_2 + \lambda_4 + \lambda_5 < 0$ which implies $\lambda_1 + \lambda_4 + \lambda_5 < 0$. Therefore we still end up with the restriction of $\lambda_1 + \lambda_4 + \lambda_5 < 0$ and do not have any nonnegative solution from this partition.

As of this submission, we have examples of lists satisfying the restrictions and that are realized by nonnegative or symmetric nonnegative matrices. Soule's construction of a symmetric nonnegative matrix with the assumption that the all one's vector is the Perron vector does not work nor does Theorem 27 as this set cannot have a reducible nonnegative matrix for a solution. This is apparent when we try to partition the positive eigenvalues with the negative ones and still meet the necessary conditions for a nonnegative solution. A detailed summary is provided in the conclusion of this chapter.

### 6.3.4 $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \lambda_4 > 0 \geqslant \lambda_5$

As $\lambda_1 + \lambda_5 \geqslant 0$, the symmetric nonnegative matrix realizing $\sigma$ is given by

$$A = \frac{1}{2}\begin{bmatrix} \lambda_1+\lambda_5 & \lambda_1-\lambda_5 & 0 & 0 & 0 \\ \lambda_1-\lambda_5 & \lambda_1+\lambda_5 & 0 & 0 & 0 \\ 0 & 0 & 2\lambda_2 & 0 & 0 \\ 0 & 0 & 0 & 2\lambda_3 & 0 \\ 0 & 0 & 0 & 0 & 2\lambda_4 \end{bmatrix}$$

### 6.3.5 $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \lambda_4 \geqslant \lambda_5 > 0$

Then $A = diag\,(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$ is a symmetric nonnegative matrix realizing $\sigma$.

## 6.4 Conclusion

There are two cases from the above where we have examples but no definite conclusions. However, in one of the cases we prove there does not exist a symmetric nonnegative matrix having the list of five real numbers as a spectrum and in the other case we have examples of symmetric nonnegative matrices with spectra satisfying the conditions. The following is a summary of the restrictions on the two cases.

1. $\lambda_1 > \lambda_2 > 0 > \lambda_3 \geqslant \lambda_4 \geqslant \lambda_5$ with restrictions

   (a) $\lambda_2 + \lambda_5 > 0$,

   (b) $\lambda_3 + \lambda_4 < \lambda_5$, and

   (c) $\lambda_1 + \lambda_3 + \lambda_4 < 0$.

2. $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 > 0 \geqslant \lambda_4 \geqslant \lambda_5$ with restrictions

   (a) $\lambda_1 > \lambda_2 \geqslant \lambda_3 > 0 > \lambda_4 \geqslant \lambda_5$,

   (b) $\lambda_2 + \lambda_5 < 0$,

   (c) $\lambda_3 + \lambda_4 < 0$, and

   (d) $\lambda_1 + \lambda_2 + \lambda_4 + \lambda_5 < 0$.

In case (1), we prove that there does not exist a solution to the SNIEP.

**Theorem 29.** *If $\sigma = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\}$ be a list of real numbers satisfying the following:*

1. $\lambda_1 > \lambda_2 > 0 > \lambda_3 \geqslant \lambda_4 \geqslant \lambda_5$,

2. $\lambda_2 + \lambda_5 > 0$,

3. $\lambda_5 > \lambda_3 + \lambda_4$, *and*

4. $\lambda_1 + \lambda_3 + \lambda_4 < 0$,

*then there does not exist a* 5 x 5 *symmetric nonnegative matrix having spectrum* $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\}$. *Moreover, the only nonnegative matrix that may realize $\sigma$ as its spectrum must be irreducible.*

*Proof.* Without loss of generality we assume the list of numbers is normalized. Then there are two cases to consider. If a nonnegative matrix realizes $\sigma$ as its spectrum, it would need to be reducible or irreducible.

1. The reducible case: Although a discussion was presented in the prior section concerning this case we repeat it here. If a nonnegative reducible matrix realizes $\sigma$, then we should be able to partition the list $\sigma$ so that each sublist would be the spectrum of a smaller nonnegative matrix. But this is impossible. As we must put two nonpositive values with one positive value for a partition to work, the smallest negative sum a positive value must dominate is $\lambda_3 + \lambda_4$ with the remaining negative value $\lambda_5$ to be dominated by the remaining positive value. The first restriction comes from $\lambda_3 + \lambda_4 \geqslant \lambda_i + \lambda_j$ for any combination of $i, j = 3, 4, 5$ with $i \neq j$. Considering the choices, we can either match the Perron root $\lambda_1$ or $\lambda_2$ with $\lambda_3 + \lambda_4$. Since $\lambda_1 \geqslant \lambda_2$ and $\lambda_2 + \lambda_5 \geqslant 0$ the best partition is $\{\lambda_1, \lambda_3, \lambda_4\}$ and $\{\lambda_2, \lambda_5\}$. But, the first list fails the trace condition as $\lambda_1 + \lambda_3 + \lambda_4 < 0$. Therefore since we can do no better than dominating the sum of $\lambda_3 + \lambda_4$ with a positive value, there cannot exist any nonnegative matrix having $\sigma = \{\lambda_1,\ \lambda_2,\ \lambda_3,\ \lambda_4,\ \lambda_5\}$ as its spectrum.

2. Suppose $\sigma$ is realized by a symmetric nonnegative matrix which is irreducible. By Lemma 1, we have $-(\lambda_5) + trace\,(A) = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 \geqslant \lambda_2$ but this implies $\lambda_1 + \lambda_3 + \lambda_4 \geqslant 0$ which contradicts the last condition in the hypothesis of this theorem. Therefore, there does not exist an irreducible symmetric nonnegative matrix realizing $\sigma$.

So for a list of five real numbers satisfying the restrictions of Theorem 29 to be the spectrum of a nonnegative matrix, the solution set of nonnegative matrices excludes

symmetric matrices and excludes reducible matrices. $\square$

The above theorem does not answer whether or not the restrictions are sufficient for there to exist a 5 x 5 irreducible, nonnegative, nonsymmetric matrix. We can prove they are not sufficient by the following examples. Consider the lists $\{5, 4, -3, -3, -3\}$ and $\{8, 7, -4, -5, -6\}$. They satisfy the conditions of Theorem 29; however, the refined JLL necessary condition that $4s_4 \geqslant s_2^2$ discussed in Chapter 5 fails as $4496 \not\geqslant 4624$ for list one and $69392 \not\geqslant 144400$ for list two. Therefore these lists cannot be spectrums of some nonnegative 5 x 5 matrix.

As discussed in Chapter 3, Laffey and Meehan [18] discovered the necessary and sufficient conditions for a list of 5 complex numbers summing to zero to be the eigenvalues of a 5 x 5 nonnegative matrix. In this particular case we have examples of lists of five real numbers for which the conditions of Theorem 29 are met and for which Theorem 11 can be applied. Consider the list $\sigma = \{97, 71, -44, -54, -70\}$ which satisfies the restrictions. For this list we get the following nonnegative matrix having $\sigma$ as its spectrum. *Mathematica* was used to create this matrix using Laffey and Meehan's construction.

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{12101}{2} + \frac{3\sqrt{418657}}{2} & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \frac{1}{60}\left(14157354960 + 20548080\sqrt{418657}\right) & 0 & 228312 & \frac{12101}{2} - \frac{3\sqrt{418657}}{2} & 0 \end{bmatrix}$$

From this case we derive the following Corollary 4.

**Corollary 4.** *In the case of a list of five real numbers, the two problems, the RNIEP and the SNIEP, are different.*

*Proof.* Consider the list $\sigma = \{97, 71, -44, -54, -70\}$. This list has trace zero and satisfies Laffey and Meehan's conditions. Hence there exists a matrix $A \geqslant 0$ such that the spectrum of $A$ is the given list.

The list also satisfies conditions of Theorem 29. Therefore there does not exist a symmetric nonnegative matrix realizing the given list as its spectrum. Hence the SNIEP and the RNIEP are different for lists of length five. $\square$

In case (2), we give examples that there do exist symmetric nonnegative matrices and hence nonnegative matrices having $\sigma = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\}$ as a spectrum. The following examples are symmetric.

$$A = \begin{bmatrix} 0 & 903 & 43 & 151 & 641 \\ 903 & 94 & 634 & 138 & 24 \\ 43 & 634 & 96 & 767 & 1 \\ 151 & 138 & 767 & 44 & 691 \\ 641 & 24 & 1 & 691 & 0 \end{bmatrix}$$

has the spectrum $\sigma = \{1657.8406, 439.1386, 342.1249, -1043.4224, -1161.6816\}$.

$$B = \begin{bmatrix} 40 & 436 & 838 & 2 & 167 \\ 436 & 0 & 0 & 177 & 730 \\ 838 & 0 & 18 & 840 & 319 \\ 2 & 177 & 840 & 97 & 851 \\ 167 & 730 & 319 & 851 & 4 \end{bmatrix}$$

has the spectrum $\sigma = \{1827.3721, 358.2611, 210.4551, -997.7358, -1239.3515\}$.

One last example of this case comes from Reams [29]. The list $\sigma = \{6, 1, 1, -4, -4\}$ satisfies all the conditions of Theorem 29 in this case and does have a trace zero nonnegative matrix having $\sigma$ as its spectrum.

**Theorem 30.** *Let $\sigma = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\}$ be a list of five real numbers satisfying the following conditions:*

1. $\lambda_2 + \lambda_5 < 0$,
2. $\lambda_3 + \lambda_4 < 0$,
3. $\lambda_1 + \lambda_2 + \lambda_4 + \lambda_5 < 0$, *and*
4. $\lambda_1 > \lambda_2 \geqslant \lambda_3 > 0 > \lambda_4 \geqslant \lambda_5$.

*A nonnegative matrix realizing $\sigma$ cannot be reducible.*

*Proof.* Let $\sigma$ satisfy the given conditions. For there to exist a nonnegative reducible matrix having $\sigma$ as its spectrum, it must be possible to partition $\sigma$ into smaller lists which are the spectra of smaller matrices. As $\lambda_1 + \lambda_5 \geqslant 0$ and we have the given

condition that $\lambda_1 + \lambda_2 + \lambda_4 + \lambda_5 < 0$, the two imply $\lambda_2 + \lambda_4 < 0$. Therefore as $\lambda_2$ does not dominate either of the negative values $\lambda_4$ or $\lambda_5$, it cannot be the Perron root for any smaller matrix. Since $\lambda_1 + \lambda_4 + \lambda_5 < 0$ we cannot partition all the negative values with $\lambda_1$. Hence if a matrix should exist with $\sigma$ as its spectrum, it would have to be irreducible. $\square$

As we have seen, this problem has many different facets to it. While the RNIEP is equivalent to the SNIEP in many cases, we now know in case (1) the two problems are indeed different. It is unclear whether or not the RNIEP and the SNIEP are different in the latter case (2). As both of the cases require irreducible matrices for possible solutions, more construction techniques or existence proofs need to be developed.

CHAPTER 7

Lists of Length Six

As we have seen in Chapter 6, the 5 x 5 case is the first case in which the RNIEP and SNIEP are different. The process of analyzing this difference came from breaking down the possible positive and nonpositive numbers in the list of five real numbers and trying to apply Kellogg's and Borobia's conditions. We continue this process on the list of six real numbers, $\sigma = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6\}$, that satisfy the following conditions. No one has looked at this scenario in the literature as of this submission.

1. $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \lambda_4 \geqslant \lambda_5 \geqslant \lambda_6$,

2. $\lambda_1 = max_{1 \leqslant i \leqslant 6} |\lambda_i|$,

3. $s_k \geqslant 0$ for $k = 1, 2, 3, \ldots$, and

4. JLL condition.

Our goal is to identify sufficient conditions that prove the existence of symmetric nonnegative matrices having the spectrum $\sigma = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6\}$.

## 7.1 Lists of Six Real Numbers

### 7.1.1 $\lambda_1 > 0 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \lambda_4 \geqslant \lambda_5 \geqslant \lambda_6$

In this case since $\lambda_1$ dominates the sum of the negative eigenvalues, we know by Suleimanova's condition that there exists a nonnegative matrix having $\sigma$ as its spectrum. Moreover, using Theorem 27 we can construct a symmetric nonnegative matrix with $\sigma$ as its spectrum.

### 7.1.2 $\lambda_1 \geqslant \lambda_2 > 0 \geqslant \lambda_3 \geqslant \lambda_4 \geqslant \lambda_5 \geqslant \lambda_6$

We begin our analysis of this by applying Kellogg's conditions which identify those elements that belong to the set $K$. In this distribution of numbers we have only the pairing of $\lambda_2 + \lambda_6$ to analyze. The possible results from this lead to three main cases with subcases hinging on the remaining Kellogg's conditions.

1. If $\lambda_2 + \lambda_6 < 0$, then we have $K = \{2\}$. The remaining Kellogg's conditions to check are that $\lambda_1 + \lambda_6 \geqslant 0$ and $\lambda_1 + (\lambda_2 + \lambda_6) + \lambda_3 + \lambda_4 + \lambda_5 \geqslant 0$. Both of these conditions must be true as the first is the Perron root condition and the second is the nonnegative trace condition. So $\lambda_2 + \lambda_6 < 0$ implies Kellogg's conditions are satisfied so by Theorem 22, we know there exists a symmetric nonnegative matrix having $\sigma$ as the spectrum.

2. If $\lambda_2 + \lambda_6 \geqslant 0$, then we have $K = \emptyset$. The remaining condition to check is whether $\lambda_1 + \lambda_3 + \lambda_4 + \lambda_5 \geqslant 0$. First we will assume this condition holds.

So for $\lambda_1 + \lambda_3 + \lambda_4 + \lambda_5 \geqslant 0$ we know by a corollary of Suleimanova's Theorem, there exists a symmetric nonnegative 4 x 4 matrix, $A_1$, having spectrum $\sigma(A_1) = \{\lambda_1, \lambda_3, \lambda_4, \lambda_5\}$. We also know there exists a symmetric nonnegative 2 x 2 matrix, $A_2$, having spectrum $\sigma(A_2) = \{\lambda_2, \lambda_6\}$ since $\lambda_2 + \lambda_6 \geqslant 0$. Putting the two together, we find the symmetric nonnegative matrix, $B$,

$$B = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$$

such that $\sigma(B) = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6\}$.

3. If $\lambda_2 + \lambda_6 \geqslant 0$ and

$$\lambda_1 + \lambda_3 + \lambda_4 + \lambda_5 < 0, \tag{3}$$

then Kellogg's conditions are not met. We now try to partition and apply Borobia's conditions. Note, to apply Borobia's conditions to any partition, we must assume the Perron root condition holds for that partition. This means $\lambda_1$ is greater than the absolute value of the most negative sum in the partition. When the Perron root condition fails for the partition, we will list it as the last subcase.

   (a) If $\lambda_1 \geqslant \lambda_2 > 0 \geqslant \lambda_6 \geqslant \lambda_3 + \lambda_4 + \lambda_5$. Then in order to apply Borobia's conditions, we must satisfy the Perron root condition, that is $\lambda_1 + \lambda_3 + \lambda_4 + \lambda_5 \geqslant 0$. But this is not possible given (3). Therefore we conclude this particular partition gives us no useful information.

As an individual partition we may gain nothing, but applying this idea to the general partition of $\lambda_1 \geqslant \lambda_2 > 0 \geqslant \lambda_i \geqslant \lambda_j + \lambda_k + \lambda_m$ where $i, j, k, m$ are distinct elements from $\{3, 4, 5, 6\}$, we are able to remove these partitions from contention. This is because $\lambda_3 + \lambda_4 + \lambda_5 \geqslant \lambda_i + \lambda_j + \lambda_k$ for any combination of the nonpositive numbers. Therefore since the Perron root condition fails when considering the most negative sum $\lambda_3 + \lambda_4 + \lambda_5$ it will fail for any other sum of three distinct nonpositive numbers also.

(b) If $\lambda_1 \geqslant \lambda_2 > 0 \geqslant \lambda_3 + \lambda_4 + \lambda_5 \geqslant \lambda_6$. Then we try to apply Kellogg's conditions to the list of four elements $\sigma_1 = \{\lambda_1, \lambda_2, \lambda_3 + \lambda_4 + \lambda_5, \lambda_6\}$. Since $\lambda_2 + \lambda_6$ is nonnegative on the original $\sigma$, we know $K = \emptyset$. However we get the next implication

$$\lambda_3 + \lambda_4 + \lambda_5 \geqslant \lambda_6 \Rightarrow$$

$$\lambda_1 + \lambda_3 + \lambda_4 + \lambda_5 \geqslant \lambda_1 + \lambda_6 \geqslant 0$$

contradicting (3). Thus this partition cannot work.

(c) If $\lambda_1 \geqslant \lambda_2 > 0 \geqslant \lambda_3 + \lambda_4 \geqslant \lambda_5 + \lambda_6$. Here we try to apply Kellogg's conditions on the set $\sigma_1 = \{\lambda_1, \lambda_2, \lambda_3 + \lambda_4, \lambda_5 + \lambda_6\}$. If $\lambda_1 + \lambda_5 + \lambda_6 \geqslant 0$ we satisfy the required Perron root condition. So assuming this is true, there are two subcases dependent on the sign of $\lambda_2 + \lambda_5 + \lambda_6$.

i. If $\lambda_2 + \lambda_5 + \lambda_6 < 0$, then $K = \{2\}$. Checking the remaining condition of $\lambda_1 + \lambda_5 + \lambda_6 \geqslant 0$ which we've assumed to be true, we have the existence of a symmetric nonnegative matrix having spectrum

$\sigma = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6\}$ by Radwan [27].

ii. If $\lambda_2 + \lambda_5 + \lambda_6 \geqslant 0$, then $K = \emptyset$. So by Suleimanova's condition and Fiedler, we know there exist symmetric nonnegative 3 x 3 matrices $A_1$ with $\sigma(A_1) = \{\lambda_1, \lambda_3, \lambda_4\}$ and $A_2$ with $\sigma(A_2) = \{\lambda_2, \lambda_5, \lambda_6\}$. Thus the symmetric matrix $B$ defined by

$$B = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$$

has spectrum $\sigma$.

iii. When the Perron root condition is not satisfied, that is $\lambda_1 + \lambda_5 + \lambda_6 < 0$, we cannot apply Radwan's result. This means the existence of a symmetric nonnegative matrix or a nonnegative matrix is unknown in this case.

(d) Suppose $\lambda_1 \geqslant \lambda_2 > 0 \geqslant \lambda_3 + \lambda_5 \geqslant \lambda_4 + \lambda_6$. Let $\sigma_1 = \{\lambda_1, \lambda_2, \lambda_3 + \lambda_5, \lambda_4 + \lambda_6\}$.

i. If $\lambda_2 + \lambda_4 + \lambda_6 < 0$, then applying Kellogg's conditions to $\sigma_1$ we have $K = \{2\}$. Then by the assumption that $\lambda_1 + \lambda_4 + \lambda_6 \geqslant 0$ all of Kellogg's conditions are met and we have the existence of a symmetric nonnegative matrix from Radwan's Theorem.

ii. If $\lambda_2 + \lambda_4 + \lambda_6 \geqslant 0$, then applying Kellogg's conditions to $\sigma_1$ we have $K = \emptyset$. Assuming $\lambda_1 + \lambda_4 + \lambda_6 \geqslant 0$, we may construct a symmetric nonnegative matrix $B$ having spectrum $\sigma$ from two smaller 3 x 3 symmetric nonnegative matrices $A_1$ and $A_2$ where $\sigma(A_1) = \{\lambda_1, \lambda_3, \lambda_5\}$

and $\sigma(A_2) = \{\lambda_2, \lambda_4, \lambda_6\}$. Here

$$B = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$$

has spectrum $\sigma$.

iii. When the Perron root condition is not satisfied, that is $\lambda_1 + \lambda_4 + \lambda_6 < 0$, we cannot apply Radwan's result. This means the existence of a symmetric nonnegative matrix or a nonnegative matrix is still unknown.

(e) Suppose $\lambda_1 \geqslant \lambda_2 > 0 \geqslant \lambda_3 + \lambda_6 \geqslant \lambda_4 + \lambda_5$. Let $\sigma_1 = \{\lambda_1, \lambda_2, \lambda_3 + \lambda_6, \lambda_4 + \lambda_5\}$.

i. If $\lambda_2 + \lambda_4 + \lambda_5 < 0$, then applying Kellogg's conditions to $\sigma_1$ we have $K = \{2\}$. Then by the assumption that $\lambda_1 + \lambda_4 + \lambda_5 \geqslant 0$ we have the existence of a symmetric nonnegative matrix from Radwan's Theorem.

ii. If $\lambda_2 + \lambda_4 + \lambda_5 \geqslant 0$, then applying Kellogg's conditions to $\sigma_1$ we have $K = \emptyset$. Assuming $\lambda_1 + \lambda_4 + \lambda_5 \geqslant 0$, we may construct a symmetric nonnegative matrix $B$ having spectrum $\sigma$ from two smaller 3 x 3 symmetric nonnegative matrices $A_1$ and $A_2$ where $\sigma(A_1) = \{\lambda_1, \lambda_3, \lambda_6\}$ and $\sigma(A_2) = \{\lambda_2, \lambda_4, \lambda_5\}$. Here

$$B = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$$

has spectrum $\sigma$.

iii. When the Perron root condition is not satisfied, that is $\lambda_1 + \lambda_4 + \lambda_5 < 0$, we cannot apply Radwan's result. This means the existence of a symmetric nonnegative matrix or a nonnegative matrix is unknown.

(f) Suppose $\lambda_1 \geqslant \lambda_2 > 0 \geqslant \lambda_4+\lambda_5 \geqslant \lambda_3+\lambda_6$. Let $\sigma_1 = \{\lambda_1, \lambda_2, \lambda_4+\lambda_5, \lambda_3+\lambda_6\}$.

i. If $\lambda_2 + \lambda_3 + \lambda_6 < 0$, then applying Kellogg's conditions to $\sigma_1$ we have $K = \{2\}$. Then by the assumption that $\lambda_1 + \lambda_3 + \lambda_6 \geqslant 0$ we have the existence of a symmetric nonnegative matrix from Radwan's Theorem.

ii. If $\lambda_2 + \lambda_3 + \lambda_6 \geqslant 0$, then applying Kellogg's conditions to $\sigma_1$ we have $K = \emptyset$. Since $\lambda_2 + \lambda_3 + \lambda_6 \geqslant 0$ it follows that $\lambda_1 + \lambda_3 + \lambda_6 \geqslant 0$, so we may construct a symmetric nonnegative matrix $B$ having spectrum $\sigma$ from two smaller 3 x 3 symmetric nonnegative matrices $A_1$ and $A_2$ where $\sigma(A_1) = \{\lambda_1, \lambda_4, \lambda_5\}$ and $\sigma(A_2) = \{\lambda_2, \lambda_3, \lambda_6\}$. Here

$$B = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$$

has spectrum $\sigma$.

iii. When the Perron root condition is not satisfied, that is $\lambda_1 + \lambda_3 + \lambda_6 < 0$, we cannot apply Radwan's result. This means the existence of a symmetric nonnegative matrix or a nonnegative matrix is unknown.

(g) Note, that to completely analyze all partitions we must consider the case where $\lambda_1 \geqslant \lambda_2 > 0 \geqslant \lambda_4 + \lambda_6 \geqslant \lambda_3 + \lambda_5$. But this is impossible given our initial assumptions that $\lambda_3 \geqslant \lambda_4$ and $\lambda_5 \geqslant \lambda_6$.

(h) Suppose $\lambda_1 \geqslant \lambda_2 > 0 \geqslant \lambda_3 \geqslant \lambda_4 \geqslant \lambda_5 + \lambda_6$.

i. If $\lambda_2 + \lambda_5 + \lambda_6 < 0$, then $K = \{2\}$ in Kellogg's conditions. Assuming $\lambda_1 + \lambda_5 + \lambda_6 \geqslant 0$ and $\lambda_1 + (\lambda_2 + \lambda_5 + \lambda_6) + \lambda_3 + \lambda_4 \geqslant 0$, there exists a symmetric nonnegative matrix having $\sigma$ as the spectrum by Radwan's Theorem.

ii. If $\lambda_2+\lambda_5+\lambda_6 \geqslant 0$, then $K = \emptyset$. Since $\lambda_2+\lambda_5+\lambda_6 \geqslant 0 \Rightarrow \lambda_1+\lambda_5+\lambda_6 \geqslant 0$, we get $\lambda_1 + \lambda_3 + \lambda_4 \geqslant \lambda_1 + \lambda_5 + \lambda_6 \geqslant 0$. This guarantees the existence of two symmetric nonnegative 3 x 3 matrices, $A_1$ and $A_2$ where $\sigma(A_1) = \{\lambda_1, \lambda_3, \lambda_4\}$ and $\sigma(A_2) = \{\lambda_2, \lambda_5, \lambda_6\}$. So we may construct the symmetric nonnegative matrix

$$B = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$$

which has the desired spectrum $\sigma$.

iii. When the Perron root condition is not satisfied, that is $\lambda_1 + \lambda_5 + \lambda_6 < 0$, we cannot apply Radwan's result. This means the existence of a symmetric nonnegative matrix or a nonnegative matrix is still unknown.

(i) Suppose $\lambda_1 \geqslant \lambda_2 > 0 \geqslant \lambda_3 \geqslant \lambda_5 \geqslant \lambda_4 + \lambda_6$.

i. If $\lambda_2 + \lambda_4 + \lambda_6 < 0$, then $K = \{2\}$. And by Radwan's Theorem, since $\lambda_1 + \lambda_4 + \lambda_6 \geqslant 0$ by assumption and $\lambda_1 + (\lambda_2 + \lambda_4 + \lambda_6) + \lambda_3 + \lambda_5 \geqslant 0$, there exists a symmetric nonnegative matrix having $\sigma$ as the spectrum.

ii. If $\lambda_2+\lambda_4+\lambda_6 \geqslant 0$, then $K=\emptyset$. Assume $\lambda_1+\lambda_4+\lambda_6 \geqslant 0$. Notice also that since $\lambda_3+\lambda_5 \geqslant \lambda_4+\lambda_6$, we have $\lambda_1+\lambda_3+\lambda_5 \geqslant \lambda_1+\lambda_4+\lambda_6 \geqslant 0$. The conditions give the existence of two symmetric nonnegative 3 x 3 matrices, $A_1$ and $A_2$ where $\sigma(A_1)=\{\lambda_1,\lambda_3,\lambda_5\}$ and $\sigma(A_2)=\{\lambda_2,\lambda_4,\lambda_6\}$. So we may construct the symmetric nonnegative matrix

$$B=\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$$

which has the desired spectrum $\sigma$.

iii. When the Perron root condition is not satisfied, that is $\lambda_1+\lambda_4+\lambda_6<0$, we cannot apply Radwan's result. This means the existence of a symmetric nonnegative matrix or a nonnegative matrix is still unknown.

(j) Suppose $\lambda_1 \geqslant \lambda_2 > 0 \geqslant \lambda_4 \geqslant \lambda_5 \geqslant \lambda_3+\lambda_6$.

i. If $\lambda_2+\lambda_3+\lambda_6<0$, then $K=\{2\}$. Assume $\lambda_1+\lambda_3+\lambda_6 \geqslant 0$. By Radwan's Theorem, since $\lambda_1+\lambda_3+\lambda_6 \geqslant 0$ and $\lambda_1+(\lambda_2+\lambda_3+\lambda_6)+\lambda_4+\lambda_5 \geqslant 0$, there exists a symmetric nonnegative matrix having $\sigma$ as the spectrum.

ii. If $\lambda_2+\lambda_3+\lambda_6 \geqslant 0$, then $K=\emptyset$. Continuing with the constraints we check if $\lambda_1+\lambda_3+\lambda_6 \geqslant 0$ and $\lambda_1+\lambda_4+\lambda_5 \geqslant 0$. The first one is true because $\lambda_1+\lambda_3+\lambda_6 \geqslant \lambda_2+\lambda_3+\lambda_6 \geqslant 0$ but the second one has no guarantee. Now we must consider the sign of $\lambda_1+\lambda_4+\lambda_5$.

If $\lambda_1+\lambda_4+\lambda_5 \geqslant 0$ then we have two symmetric nonnegative 3 x 3 matrices, $A_1$ and $A_2$ where $\sigma(A_1) = \{\lambda_1, \lambda_4, \lambda_5\}$ and $\sigma(A_2) = \{\lambda_2, \lambda_3, \lambda_6\}$. So we may construct the symmetric nonnegative matrix

$$B = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$$

which has the desired spectrum $\sigma$.

If $\lambda_1 + \lambda_4 + \lambda_5 < 0$ we cannot apply Radwan's result. This means the existence of a symmetric nonnegative matrix or a nonnegative matrix is still unknown.

iii. When the Perron root condition is not satisfied, that is $\lambda_1 + \lambda_3 + \lambda_6 < 0$, we cannot apply Radwan's result. This means the existence of a symmetric nonnegative matrix or a nonnegative matrix is still unknown.

(k) Suppose $\lambda_1 \geqslant \lambda_2 > 0 \geqslant \lambda_3 \geqslant \lambda_6 \geqslant \lambda_4 + \lambda_5$.

i. If $\lambda_2 + \lambda_4 + \lambda_5 < 0$, then $K = \{2\}$. By Radwan's Theorem, since $\lambda_1 + \lambda_4 + \lambda_5 \geqslant 0$ by assumption and $\lambda_1 + (\lambda_2 + \lambda_4 + \lambda_5) + \lambda_3 + \lambda_6 \geqslant 0$, there exists a symmetric nonnegative matrix having $\sigma$ as the spectrum.

ii. If $\lambda_2 + \lambda_4 + \lambda_5 \geqslant 0$, then $K = \emptyset$. Continuing with the constraints we check if $\lambda_1 + \lambda_4 + \lambda_5 \geqslant 0$ and $\lambda_1 + \lambda_3 + \lambda_6 \geqslant 0$. The first one is true because $\lambda_1 + \lambda_4 + \lambda_5 \geqslant \lambda_2 + \lambda_4 + \lambda_5 \geqslant 0$ but the second one has no guarantee. Now we must consider the sign of $\lambda_1 + \lambda_3 + \lambda_6$.

If $\lambda_1+\lambda_3+\lambda_6 \geqslant 0$ then we have two symmetric nonnegative 3 x 3 matrices, $A_1$ and $A_2$ where $\sigma(A_1) = \{\lambda_1, \lambda_3, \lambda_6\}$ and $\sigma(A_2) = \{\lambda_2, \lambda_4, \lambda_5\}$. So we may construct the symmetric nonnegative matrix

$$B = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$$

which has the desired spectrum $\sigma$.

If $\lambda_1 + \lambda_3 + \lambda_6 < 0$ we cannot apply Radwan's result. This means the existence of a symmetric nonnegative matrix or a nonnegative matrix is still unknown.

iii. When the Perron root condition is not satisfied, that is $\lambda_1 + \lambda_4 + \lambda_5 < 0$, we cannot apply Radwan's result. This means the existence of a symmetric nonnegative matrix or a nonnegative matrix is still unknown.

(l) Suppose $\lambda_1 \geqslant \lambda_2 > 0 \geqslant \lambda_3 \geqslant \lambda_4 + \lambda_5 \geqslant \lambda_6$.

i. If $\lambda_2 + \lambda_6 < 0$, then $K = \{2\}$. Our next conditions to satisfy are $\lambda_1 + \lambda_6 \geqslant 0$ and $\lambda_1 + (\lambda_2 + \lambda_6) + \lambda_3 + \lambda_4 + \lambda_5 \geqslant 0$. Both of these are true. The first is because of the Perron root condition on $\sigma$. The second is the nonnegative trace condition. Here Radwan's Theorem applies and we have the existence of a symmetric nonnegative matrix having $\sigma$ as the spectrum.

ii. If $\lambda_2 + \lambda_6 \geqslant 0$, then $K = \emptyset$. Checking the next condition of Kellogg, we see $\lambda_1 + \lambda_3 + \lambda_4 + \lambda_5 \ngeqslant 0$ because of (3).

Hence, this particular partition gains no further restrictions for the existence of symmetric nonnegative matrices or nonnegative matrices. We do however point out that no other partition $\lambda_1 \geqslant \lambda_2 > 0 \geqslant \lambda_i \geqslant \lambda_j + \lambda_k \geqslant \lambda_6$ works for $i, j, k$ being nonrepeating elements from $\{3, 4, 5\}$.

(m) Suppose $\lambda_1 \geqslant \lambda_2 > 0 \geqslant \lambda_3 + \lambda_4 \geqslant \lambda_5 \geqslant \lambda_6$.

i. If $\lambda_2 + \lambda_6 < 0$, then $K = \{2\}$. Our next conditions to satisfy are $\lambda_1 + \lambda_6 \geqslant 0$ and $\lambda_1 + (\lambda_2 + \lambda_6) + \lambda_3 + \lambda_4 + \lambda_5 \geqslant 0$. Both of these are true. The first is because of the Perron root condition on $\sigma$. The second is the nonnegative trace condition. Here Radwan's result applies and we have the existence of a symmetric nonnegative matrix having $\sigma$ as the spectrum.

ii. If $\lambda_2 + \lambda_6 \geqslant 0$, then $K = \emptyset$. Checking the next condition of Kellogg, we see $\lambda_1 + \lambda_3 + \lambda_4 + \lambda_5 \ngeqslant 0$ because of (3). Hence, this particular partition gains no further restrictions for the existence of symmetric nonnegative matrices or nonnegative matrices.

### 7.1.3 $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 > 0 \geqslant \lambda_4 \geqslant \lambda_5 \geqslant \lambda_6$

In this particular distribution of numbers we find many more cases because we have more positive values to match up with negative ones. For example, the straight

application of Kellogg's conditions requires us to analyze $\lambda_2 + \lambda_6$ and $\lambda_3 + \lambda_5$ to determine whether or not $2, 3 \in K$. We will consider all possible outcomes through the following.

1. If

$$\lambda_2 + \lambda_6 < 0 \tag{4}$$

and

$$\lambda_3 + \lambda_5 < 0, \tag{5}$$

then $K = \{2, 3\}$. We will examine the different outcomes of the remaining conditions of Kellogg, that is $\lambda_1 + \lambda_6 \geqslant 0$, $\lambda_1 + (\lambda_2 + \lambda_6) + \lambda_5 \geqslant 0$, and $\lambda_1 + (\lambda_2 + \lambda_6) + (\lambda_3 + \lambda_5) + \lambda_4 \geqslant 0$. The first and third are true by the necessary conditions on $\sigma$ with regard to the Perron root condition and the nonnegative trace condition. The second condition need not be true.

   (a) If $\lambda_1 + (\lambda_2 + \lambda_6) + \lambda_5 \geqslant 0$, then as we satisfy all of Kellogg's conditions, we have by Fiedler's result a symmetric nonnegative matrix.

   (b) If

$$\lambda_1 + (\lambda_2 + \lambda_6) + \lambda_5 < 0, \tag{6}$$

   then Kellogg's conditions are not satisfied. Note that $\lambda_1 + (\lambda_2 + \lambda_6) + \lambda_5 < 0$ gives

$$\lambda_3 + \lambda_4 \geqslant 0 \tag{7}$$

due to the nonnegative trace requirement. In this case where Kellogg's conditions are not met, we try partitioning to use Borobia's and Radwan's results to further resolve the problem.

i. If $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 > 0 \geqslant \lambda_4 \geqslant \lambda_5 + \lambda_6$. Then to make this partition viable, we must have $\lambda_1 + \lambda_5 + \lambda_6 \geqslant 0$. But this clearly contradicts (6) as $\lambda_2 > 0$. Therefore this partition fails to give new information.

ii. If $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 > 0 \geqslant \lambda_5 \geqslant \lambda_4 + \lambda_6$, then the following cases analyze the possible K sets for Borobia's conditions.

   A. If $\lambda_2 + \lambda_4 + \lambda_6 < 0$ and $\lambda_3 + \lambda_5 < 0$, then $K = \{2, 3\}$. Then Borobia's and Radwan's conditions are satisfied if $\lambda_1 + \lambda_4 + \lambda_6 \geqslant 0$ and $\lambda_1 + (\lambda_2 + \lambda_4 + \lambda_6) + \lambda_5 \geqslant 0$. The condition $\lambda_1 + \lambda_4 + \lambda_6 \geqslant 0$ may or may not be true. For this case we will assume it is, but we will address when it does not in the final case of this partition. But the second condition $\lambda_1 + (\lambda_2 + \lambda_4 + \lambda_6) + \lambda_5 \geqslant 0$ cannot be true as it contradicts (6). Therefore this partition when $K = \{2, 3\}$ cannot happen.

   B. If $\lambda_2 + \lambda_4 + \lambda_6 < 0$ and $\lambda_3 + \lambda_5 \geqslant 0$, then $K = \{2\}$. But $\lambda_3 + \lambda_5 \geqslant 0$ contradicts (5). Therefore this partition cannot happen.

   C. If $\lambda_2 + \lambda_4 + \lambda_6 \geqslant 0$ and $\lambda_3 + \lambda_5 < 0$, then $K = \{3\}$. But $\lambda_2 + \lambda_4 + \lambda_6 \geqslant 0$ contradicts (4). Therefore this partition cannot happen.

   D. If $\lambda_2 + \lambda_4 + \lambda_6 \geqslant 0$ and $\lambda_3 + \lambda_5 \geqslant 0$, then $K = \emptyset$. But again, each

condition contradicts either (4) or (5). Therefore this partition cannot happen.

E. If $\lambda_1 + \lambda_4 + \lambda_6 < 0$, then the Perron root condition is not satisfied in this partition. Then this partition does not resolve whether or not there exists a symmetric nonnegative matrix or a nonnegative matrix having $\sigma$ as a spectrum.

iii. If $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 > 0 \geqslant \lambda_6 \geqslant \lambda_4 + \lambda_5$, then the following cases analyze the possible K sets for Borobia's conditions.

A. If $\lambda_2 + \lambda_4 + \lambda_5 < 0$ and $\lambda_3 + \lambda_6 < 0$, then $K = \{2, 3\}$. Then Borobia's and Radwan's conditions are satisfied if $\lambda_1 + \lambda_4 + \lambda_5 \geqslant 0$ and $\lambda_1 + (\lambda_2 + \lambda_4 + \lambda_5) + \lambda_6 \geqslant 0$. The condition $\lambda_1 + \lambda_4 + \lambda_5 \geqslant 0$ may or may not be true. For this case we will assume it is, but we will address when it does not in the final case of this partition. But the second condition $\lambda_1 + (\lambda_2 + \lambda_4 + \lambda_5) + \lambda_6 \geqslant 0$ cannot be true as it contradicts (6). Therefore this partition when $K = \{2, 3\}$ cannot happen.

B. If $\lambda_2 + \lambda_4 + \lambda_5 < 0$ and $\lambda_3 + \lambda_6 \geqslant 0$, then $K = \{2\}$. But $\lambda_3 + \lambda_6 \geqslant 0$ contradicts (5). Therefore this partition cannot happen.

C. If $\lambda_2 + \lambda_4 + \lambda_5 \geqslant 0$ and $\lambda_3 + \lambda_6 < 0$, then $K = \{3\}$. But $\lambda_2 + \lambda_4 + \lambda_5 \geqslant 0$ contradicts $0 > \lambda_2 + \lambda_6 \geqslant \lambda_2 + \lambda_4 + \lambda_5$ using (4) and the partition (iii). Therefore this partition cannot happen.

D. If $\lambda_2 + \lambda_4 + \lambda_5 \geqslant 0$ and $\lambda_3 + \lambda_6 \geqslant 0$, then $K = \emptyset$. But again, each condition contradicts known prior conditions. Therefore this partition cannot happen.

E. If $\lambda_1 + \lambda_4 + \lambda_5 < 0$, then the Perron root condition is not satisfied with this partition. Then this partition does not resolve whether or not there exists a symmetric nonnegative matrix or a nonnegative matrix having $\sigma$ as a spectrum.

iv. If $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 > 0 \geqslant \lambda_4 + \lambda_5 \geqslant \lambda_6$, then we show that the only viable case for $K$ is when $K = \{2, 3\}$. This means the inequalities $\lambda_2 + \lambda_6 < 0$ and $\lambda_3 + \lambda_4 + \lambda_5 < 0$ are true. We know they must be because of (4) and (5). These conditions rule out any other possible $K$. So in this only viable case, for the remaining conditions to be true we need $\lambda_1 + \lambda_6 \geqslant 0$ and $\lambda_1 + (\lambda_2 + \lambda_6) + \lambda_4 + \lambda_5 \geqslant 0$. Unfortunately the second one contradicts (6). Therefore even this case fails.

v. If $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 > 0 \geqslant \lambda_4 + \lambda_5 + \lambda_6$, then to discuss Borobia's and Radwan's conditions we must have $\lambda_1 + \lambda_4 + \lambda_5 + \lambda_6 \geqslant 0$. But clearly this contradicts (6), so this partition gains us nothing.

The previous partitions were based on the original distribution of $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 > 0 \geqslant \lambda_4 \geqslant \lambda_5 \geqslant \lambda_6$ where

$$\lambda_2 + \lambda_6 < 0,$$

$$\lambda_3 + \lambda_5 < 0,$$

$$\lambda_1 + (\lambda_2 + \lambda_6) + \lambda_5 < 0, \text{ and}$$

$$\lambda_3 + \lambda_4 \geqslant 0.$$

2. Let us now address the case where we have $K = \{2\}$ with regard to the original spread of $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 > 0 \geqslant \lambda_4 \geqslant \lambda_5 \geqslant \lambda_6$. If

$$\lambda_2 + \lambda_6 < 0 \tag{8}$$

and

$$\lambda_3 + \lambda_5 \geqslant 0, \tag{9}$$

then $K = \{2\}$. We will examine the different outcomes of the remaining conditions of Kellogg, that is $\lambda_1 + \lambda_6 \geqslant 0$, and $\lambda_1 + (\lambda_2 + \lambda_6) + \lambda_4 \geqslant 0$. The first is true by the necessary condition on $\sigma$ with regard to the Perron root condition. The second condition need not be true. We will analyze this in the following.

   (a) If $\lambda_1 + \lambda_6 \geqslant 0$ and $\lambda_1 + (\lambda_2 + \lambda_6) + \lambda_4 \geqslant 0$, then Kellogg's Theorem applies and by Fiedler's result we know there exists a symmetric nonnegative matrix having $\sigma$ as the spectrum.

   (b) If

$$\lambda_1 + (\lambda_2 + \lambda_6) + \lambda_4 < 0, \tag{10}$$

Kellogg's conditions fail. This leaves us with Borobia's and Radwan's results which allow for partitions. The following cases consider all possible partitions.

i. If $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 > 0 \geqslant \lambda_6 \geqslant \lambda_4 + \lambda_5$, then consider $\sigma_1 = \{\lambda_1, \lambda_2, \lambda_3, \lambda_6, \lambda_4 + \lambda_5\}$. We will analyze the possible $K$ sets from applying the conditions to $\sigma_1$.

   A. If $\lambda_2 + \lambda_4 + \lambda_5 < 0$ and $\lambda_3 + \lambda_6 < 0$, then $K = \{2, 3\}$. Combine this with the Perron root condition by assuming that $\lambda_1 + \lambda_4 + \lambda_5 \geqslant 0$, then we need only consider the inequality $\lambda_1 + (\lambda_2 + \lambda_4 + \lambda_5) + \lambda_6 \geqslant 0$. Unfortunately this final inequality contradicts (10). Therefore this case cannot happen.

   B. If $\lambda_2 + \lambda_4 + \lambda_5 < 0$ and $\lambda_3 + \lambda_6 \geqslant 0$, then $K = \{2\}$. But $\lambda_3 + \lambda_6 \geqslant 0$ contradicts (8), so this case cannot happen.

   C. If $\lambda_2 + \lambda_4 + \lambda_5 \geqslant 0$ then using (8) and the partition we get $0 > \lambda_2 + \lambda_6 \geqslant \lambda_2 + \lambda_4 + \lambda_5$ which is a contradiction. So we have ruled out $K = \{3\}$ and $K = \emptyset$.

   D. If $\lambda_1 + \lambda_4 + \lambda_5 < 0$, then we do not satisfy the Perron root condition with this partition. Therefore we do not know whether or not there exists a symmetric nonnegative matrix or a nonnegative matrix having $\sigma$ as a spectrum.

ii. If $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 > 0 \geqslant \lambda_4 \geqslant \lambda_5 + \lambda_6$, then consider $\sigma_1 = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 + \lambda_6\}$. But this set fails the Perron root condition, that is $\lambda_1 + \lambda_5 + \lambda_6 \geqslant 0$ because it contradicts (10). Therefore, this partition does not give us anything new.

iii. If $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 > 0 \geqslant \lambda_5 \geqslant \lambda_4 + \lambda_6$. Then consider $\sigma_1 = \{\lambda_1, \lambda_2, \lambda_3, \lambda_5, \lambda_4 + \lambda_6\}$. We will analyze the possible $K$ sets from applying Kellogg's conditions to $\sigma_1$.

A. If $\lambda_2 + \lambda_4 + \lambda_6 < 0$ and $\lambda_3 + \lambda_5 < 0$, then $K = \{2, 3\}$. Combine this with the Perron root condition by assuming that $\lambda_1 + \lambda_4 + \lambda_6 \geqslant 0$, then we need only consider the inequality $\lambda_1 + (\lambda_2 + \lambda_4 + \lambda_6) + \lambda_5 \geqslant 0$. Unfortunately this final inequality contradicts (10). Therefore this case cannot happen.

B. If $\lambda_2 + \lambda_4 + \lambda_6 < 0$ and $\lambda_3 + \lambda_5 \geqslant 0$, then $K = \{2\}$. But to apply Borobia's and Radwan's conditions, we need $\lambda_1 + \lambda_4 + \lambda_6 \geqslant 0$. This inequality contradicts (10) so this case cannot occur.

C. If $\lambda_2 + \lambda_4 + \lambda_6 \geqslant 0$, then we contradict (8). It follows that $K = \{3\}$ and $K = \emptyset$ cannot occur.

D. If $\lambda_1 + \lambda_4 + \lambda_6 < 0$, then we do not satisfy the Perron root condition with this partition. Therefore we do not know whether or not there exists a symmetric nonnegative matrix or a nonnegative matrix having $\sigma$ as a spectrum.

iv. If $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 > 0 \geqslant \lambda_4 + \lambda_5 \geqslant \lambda_6$, then consider $\sigma_1 = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4 + \lambda_5, \lambda_6\}$. We will analyze the possible $K$ sets from applying Kellogg's conditions to $\sigma_1$.

A. If $\lambda_2 + \lambda_6 < 0$ and $\lambda_3 + \lambda_4 + \lambda_5 < 0$, then $K = \{2, 3\}$. Here the necessary Perron root condition is met since that $\lambda_1 + \lambda_6 \geqslant 0$ for $\sigma$. Hence we need only consider the inequality $\lambda_1 + (\lambda_2 + \lambda_6) + \lambda_4 + \lambda_5 \geqslant 0$. Unfortunately this final inequality contradicts (10). Therefore this case cannot happen.

B. If $\lambda_2 + \lambda_6 < 0$ and $\lambda_3 + \lambda_4 + \lambda_5 \geqslant 0$, then $K = \{2\}$. Then we may subdivide $\sigma_1$ into $\sigma_2 = \{\lambda_1, \lambda_6\}$, $\sigma_3 = \{\lambda_3, \lambda_4, \lambda_5\}$, and $\sigma_4 = \{\lambda_2\}$. Each of these lists satisfy Suleimanova's condition to be the spectrums of symmetric nonnegative matrices, $A_2$, $A_3$, and $A_4$ respectively. Therefore a symmetric nonnegative matrix having $\sigma$ as its spectrum is

$$B = \begin{bmatrix} A_2 & 0 & 0 \\ 0 & A_3 & 0 \\ 0 & 0 & A_4 \end{bmatrix}.$$

C. If $\lambda_2 + \lambda_6 \geqslant 0$ it contradicts (8). So it follows that $K = \{3\}$ and $K = \emptyset$ cannot occur.

D. If $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 > 0 \geqslant \lambda_4 + \lambda_5 + \lambda_6$. Then the only possible grouping under question is $\lambda_2 + \lambda_4 + \lambda_5 + \lambda_6$. As we must satisfy (10),

this expression must be negative, that is $\lambda_2 + \lambda_4 + \lambda_5 + \lambda_6 < 0$. Therefore the only element in the $K$ set for this partition is 2. As $\lambda_1 + \lambda_4 + \lambda_5 + \lambda_6 < 0$ because of (10), we find the Perron root condition fails for this partition. Therefore this partition gives no new information to the problem.

The previous partitions were based on the original distribution of six real numbers where

$$\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 > 0 \geqslant \lambda_4 \geqslant \lambda_5 \geqslant \lambda_6,$$

$$\lambda_2 + \lambda_6 < 0,$$

$$\lambda_3 + \lambda_5 \geqslant 0,$$

$$\lambda_1 + (\lambda_2 + \lambda_6) + \lambda_4 < 0, \text{ and}$$

$$\lambda_3 + \lambda_5 \geqslant 0.$$

3. Let us now address the case where we have $K = \{3\}$ with regard to the original spread of $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 > 0 \geqslant \lambda_4 \geqslant \lambda_5 \geqslant \lambda_6$. If

$$\lambda_2 + \lambda_6 \geqslant 0 \tag{11}$$

and

$$\lambda_3 + \lambda_5 < 0, \tag{12}$$

then $K = \{3\}$. We will examine the different outcomes of the remaining conditions of Kellogg, that is $\lambda_1 + \lambda_5 \geqslant 0$, and $\lambda_1 + (\lambda_3 + \lambda_5) + \lambda_4 \geqslant 0$. The first is true by the Perron root condition on $\sigma$. The second condition need not be true. We will analyze this in the following.

(a) If $\lambda_1 + \lambda_5 \geqslant 0$ and $\lambda_1 + (\lambda_3 + \lambda_5) + \lambda_4 \geqslant 0$, then Kellogg's Theorem applies and by Fiedler's result we know there exists a symmetric nonnegative matrix having $\sigma$ as the spectrum.

(b) If

$$\lambda_1 + (\lambda_3 + \lambda_5) + \lambda_4 < 0, \tag{13}$$

then Kellogg's conditions fail. This leaves us with Borobia's and Radwan's results which require partitions. The following cases consider all possible partitions.

i. If $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 > 0 \geqslant \lambda_4 \geqslant \lambda_5 + \lambda_6$. This partition does not apply as the Perron root condition fails, that is $\lambda_1 + (\lambda_5 + \lambda_6) \ngeqslant 0$ by (13).

ii. If $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 > 0 \geqslant \lambda_5 \geqslant \lambda_4 + \lambda_6$. This partition does not apply as the Perron root condition fails, that is $\lambda_1 + (\lambda_4 + \lambda_6) \ngeqslant 0$ by (13).

iii. If $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 > 0 \geqslant \lambda_6 \geqslant \lambda_4 + \lambda_5$. This partition does not apply as the Perron root condition fails, that is $\lambda_1 + (\lambda_4 + \lambda_5) \ngeqslant 0$ by (13).

iv. If $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 > 0 \geqslant \lambda_4 + \lambda_5 \geqslant \lambda_6$. This partition cannot occur because $\lambda_1 + \lambda_4 + \lambda_5 \geqslant \lambda_1 + \lambda_6 \geqslant 0$ is a contradiction to (13).

v. If $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 > 0 \geqslant \lambda_4 + \lambda_5 + \lambda_6$. This partition cannot occur given the restriction from (13). That is $0 > \lambda_1 + \lambda_2 + \lambda_4 + \lambda_5 > \lambda_1 + \lambda_4 + \lambda_5 + \lambda_6$ fails the Perron root condition.

The previous partitions were based on the original distribution of six number where

$$\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 > 0 \geqslant \lambda_4 \geqslant \lambda_5 \geqslant \lambda_6,$$

$$\lambda_2 + \lambda_6 \geqslant 0,$$

$$\lambda_3 + \lambda_5 < 0, \text{ and}$$

$$\lambda_1 + (\lambda_3 + \lambda_5) + \lambda_4 < 0.$$

4. Let us now address the case where we have $K = \emptyset$ with regard to the original spread of $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 > 0 \geqslant \lambda_4 \geqslant \lambda_5 \geqslant \lambda_6$. For $K = \emptyset$ then we must have $\lambda_2 + \lambda_6 \geqslant 0$, and $\lambda_3 + \lambda_5 \geqslant 0$. But since $\lambda_1 + \lambda_4 \geqslant 0$ we satisfy the sufficient conditions necessary for there to exist three 2 x 2 symmetric nonnegative matrices, $A_1$, $A_2$, and $A_3$ having spectrums $\sigma(A_1) = \{\lambda_1, \lambda_4\}$, $\sigma(A_2) = \{\lambda_2, \lambda_6\}$, and $\sigma(A_3) = \{\lambda_3, \lambda_5\}$. So we may construct the symmetric nonnegative matrix

$$B = \begin{bmatrix} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_3 \end{bmatrix}$$

realizing $\sigma$ as its spectrum.

### 7.1.4 $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \lambda_4 > 0 \geqslant \lambda_5 \geqslant \lambda_6$

In this particular distribution the straight application of Kellogg's conditions requires us to analyze $\lambda_2 + \lambda_6$ and $\lambda_3 + \lambda_5$ to determine whether or not $2, 3 \in K$. We will consider all possible outcomes through the following.

1. If

$$\lambda_2 + \lambda_6 < 0 \tag{14}$$

and

$$\lambda_3 + \lambda_5 < 0, \tag{15}$$

then $K = \{2,3\}$. We will examine the different outcomes of the remaining conditions of Kellogg, that is $\lambda_1 + \lambda_6 \geqslant 0$, $\lambda_1 + (\lambda_2 + \lambda_6) + \lambda_5 \geqslant 0$, and $\lambda_1 + (\lambda_2 + \lambda_6) + (\lambda_3 + \lambda_5) \geqslant 0$. The first is true by the necessary condition on $\sigma$ with regard to the Perron root condition. The second condition need not be true. The third condition depends on the second. Notice that if $\lambda_1 + (\lambda_2 + \lambda_6) + \lambda_5 \geqslant 0$, then $\lambda_1 + (\lambda_2 + \lambda_6) + (\lambda_3 + \lambda_5) < 0$ cannot happen for $\lambda_3 \geqslant 0$. So we may conclude the only viable condition is the second one, that is $\lambda_1 + (\lambda_2 + \lambda_6) + \lambda_5 \geqslant 0$. If this condition is satisfied there is a symmetric nonnegative matrix whose spectrum is $\sigma$.

Suppose

$$\lambda_1 + (\lambda_2 + \lambda_6) + \lambda_5 < 0, \tag{16}$$

we find that $\sigma$ does not satisfy the Kellogg conditions. Let us consider the following partition to see if this distribution may be solved using Borobia's and Radwan's conditions. If $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \lambda_4 > 0 \geqslant \lambda_5 + \lambda_6$. This partition does not apply as the Perron root condition fails, that is $\lambda_1 + \lambda_5 + \lambda_6 < 0$ because of (16).

2. If

$$\lambda_2 + \lambda_6 < 0 \tag{17}$$

and

$$\lambda_3 + \lambda_5 \geqslant 0, \tag{18}$$

then $K = \{2\}$. Again, using the Perron root condition from $\sigma$ and (16) we can find symmetric nonnegative matrices, $A_1$ and $A_2$ such that $\sigma(A_1) = \{\lambda_1, \lambda_6\}$ and $\sigma(A_2) = \{\lambda_3, \lambda_5\}$. Using these, we can construct the symmetric nonnegative matrix

$$B = \begin{bmatrix} A_1 & 0 & 0 & 0 \\ 0 & A_2 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix}$$

having spectrum $\sigma$.

3. If

$$\lambda_2 + \lambda_6 \geqslant 0 \tag{19}$$

and

$$\lambda_3 + \lambda_5 < 0, \tag{20}$$

then $K = \{3\}$. Again, using the Perron root condition from $\sigma$ and (19) we can find symmetric nonnegative matrices, $A_1$ and $A_2$ such that $\sigma(A_1) = \{\lambda_1, \lambda_5\}$ and $\sigma(A_2) = \{\lambda_2, \lambda_6\}$. Using these, we can construct the symmetric nonnegative

matrix

$$B = \begin{bmatrix} A_1 & 0 & 0 & 0 \\ 0 & A_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix}$$

having spectrum $\sigma$.

4. If

$$\lambda_2 + \lambda_6 \geqslant 0 \tag{21}$$

and

$$\lambda_3 + \lambda_5 \geqslant 0, \tag{22}$$

then $K = \emptyset$. Again, using (21) and (22) we can find symmetric nonnegative matrices, $A_1$ and $A_2$ such that $\sigma(A_1) = \{\lambda_2, \lambda_6\}$ and $\sigma(A_2) = \{\lambda_3, \lambda_5\}$. Therefore, we can construct the symmetric nonnegative matrix

$$B = \begin{bmatrix} A_1 & 0 & 0 & 0 \\ 0 & A_2 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix}$$

having spectrum $\sigma$.

### 7.1.5 $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \lambda_4 \geqslant \lambda_5 > 0 \geqslant \lambda_6$

In this particular distribution we find a simple solution from the necessary Perron root condition, that is $\lambda_1 + \lambda_6 \geqslant 0$. With $\lambda_1, \lambda_6$ as the eigenvalues of a 2 x 2 symmetric

nonnegative matrix, we can easily find a symmetric nonnegative matrix having $\sigma$ as its spectrum. The following is one such matrix.

$$\begin{bmatrix} \frac{\lambda_1+\lambda_6}{2} & \frac{\lambda_1-\lambda_6}{2} & 0 & 0 & 0 & 0 \\ \frac{\lambda_1-\lambda_6}{2} & \frac{\lambda_1+\lambda_6}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_5 \end{bmatrix}$$

## 7.2 Additional Results

In chapter 6 we have a useful result from McDonald and Neumann [21] in regard to an irreducible 5 x 5 nonnegative symmetric matrix having eigenvalues $1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \lambda_4 \geqslant \lambda_5 \geqslant -1$ where 1 is the Perron root. A similar result is obtained in higher dimensions. We give a proof in the same lines of McDonald and Newmann for a list of six real numbers and then generalize it to a list of $n$ real numbers.

**Theorem 31.** *Let $A$ be a* 6 x 6 *irreducible nonnegative symmetric matrix with eigenvalues* $\{1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6\}$ *where* $1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \lambda_4 \geqslant \lambda_5 \geqslant \lambda_6 \geqslant -1$. *Then* $-2\lambda_6 + trace\,(A) \geqslant \lambda_2$.

*Moreover, if some eigenvector of $A$ corresponding to $\lambda_2$ has*

1. *exactly two positive or exactly two negative entries, then* $-\lambda_6 + trace\,(A) \geqslant \lambda_2$

   *or*

2. *exactly one positive or exactly one negative entry, then* $trace\,(A) \geqslant \lambda_2$.

*Proof.* Let $A = [a_{ij}]$ where $i, j = 1, 2, 3, 4, 5, 6$, be an irreducible nonnegative symmetric matrix with eigenvalues $\{1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6\}$ where $1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \lambda_4 \geqslant \lambda_5 \geqslant \lambda_6 \geqslant -1$. First suppose $\lambda_6 \geqslant 0$. Then $-\lambda_6 + trace\,(A) = -\lambda_6 + \sum_{i=1}^{6} \lambda_i = 1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 \geqslant \lambda_2$, since $\lambda_i \geqslant 0$ for $i = 1, 2, \ldots, 6$.

Now suppose $\lambda_6 < 0$. Then let $B = A - \lambda_6 I$. $B$ is positive semidefinite since $B$ is symmetric and has nonnegative eigenvalues, that is $\sigma\,(B) = \{1 - \lambda_6, \lambda_2 - \lambda_6, \lambda_3 - \lambda_6, \lambda_4 - \lambda_6, \lambda_5 - \lambda_6, 0\}$. And since $A$ is nonnegative, we find that $A - \lambda_6 I = B$ is also nonnegative. Therefore $B$ is positive semidefinite, nonnegative, and symmetric.

Let $x$ be the Perron vector corresponding to the Perron root $1 - \lambda_6$ for $B$. By the Perron-Frobenius Theorem $x > 0$. So we may choose $x$ such that $\|x\|_2 = 1$. Let $y$ be an eigenvector corresponding to $\lambda_2 - \lambda_6$. Since $B$ is symmetric, $x^T y = 0$ so we may conclude that $y$ has at least one negative entry and at least one positive entry. As $y$ has six entries, then either $y$ or $-y$ has at most three positive entries. Without loss of generality let $y$ contain at most three positive entries positioned at the top

of the vector. Let $y_1$ be the positive part of $y$, and $y_2$ be the nonpositive part of $y$. Therefore

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

Partition $B$ to match $y$. So when $y_1$ is 3 x 1 then

$$\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} B_{11}y_1 + B_{12}y_2 \\ B_{21}y_1 + B_{22}y_2 \end{bmatrix}$$

where $B_{11}$ is 3 x 3 and $B_{22}$ is 3 x 3.

Then $B_{11}y_1 + B_{12}y_2 = (\lambda_2 - \lambda_6)\, y_1$ and since $B_{12}y_2 < 0$ it follows that $B_{11}y_1 \geqslant (\lambda_2 - \lambda_6)\, y_1$. Now as $B$ is nonnegative and semidefinite, so is $B_{11}$, therefore as

$$B_{11} = \begin{bmatrix} a_{11} - \lambda_6 & a_{12} & a_{13} \\ a_{21} & a_{22} - \lambda_6 & a_{23} \\ a_{31} & a_{32} & a_{33} - \lambda_6 \end{bmatrix},$$

we have

$$\begin{aligned} trace\,(B_{11}) &= a_{11} + a_{22} + a_{33} - 3\lambda_6 \\ &\leqslant a_{11} + a_{22} + a_{33} + a_{44} + a_{55} - 3\lambda_6 \\ &= trace\,(A) - 3\lambda_6. \end{aligned}$$

Now the Perron root $\rho\,(B_{11})$ satisfies $\rho\,(B_{11}) \geqslant \lambda_2 - \lambda_6$. Therefore $-3\lambda_6 + trace\,(A) \geqslant trace\,(B_{11}) \geqslant \rho\,(B_{11}) \geqslant \lambda_2 - \lambda_6$ giving our desired conclusion $-2\lambda_6 + trace\,(A) \geqslant \lambda_2$.

A similar process applies when $y_1$ is 2 x 1 or 1 x 1. In the first case, this implies $B_{11}$ is a 2 x 2 matrix and $B_{22}$ is a 4 x 4 matrix, so $B_{11}y_1 \geqslant (\lambda_2 - \lambda_6)\, y_1$ so we have $-2\lambda_6 + trace\,(A) \geqslant trace\,(B_{11}) \geqslant \rho\,(B_{11}) \geqslant \lambda_2 - \lambda_6$ giving $-\lambda_6 + trace\,(A) \geqslant \lambda_2$.

When $y_1$ is 1 x 1, then $-\lambda_6 + trace\,(A) \geqslant trace\,(B_{11}) \geqslant \rho\,(B_{11}) \geqslant \lambda_2 - \lambda_6$. Therefore, $trace\,(A) \geqslant \lambda_2$. □

An easy generalization of the previous result is

**Theorem 32.** *Let $A$ be an $n$ x $n$ irreducible nonnegative symmetric matrix with eigenvalues $\{1, \lambda_2, \ldots, \lambda_n\}$ where $1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n \geqslant -1$, then $(4-n)\,\lambda_n + trace\,(A) \geqslant \lambda_2$.*

Another simple result uses the ideas from Perfect and Reams [29]. Let $\lambda_1, \lambda_2, \ldots, \lambda_6$ be the eigenvalues of a 6 x 6 matrix, $A$. We may represent by $p_i$ the sum of all $i^{th}$ products of the eigenvalues as follows:

1. $p_1 = \sum_{i=1}^{6} \lambda_i$,
2. $p_2 = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_1\lambda_5 + \lambda_1\lambda_6 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_2\lambda_5 + \lambda_2\lambda_6 + \lambda_3\lambda_4 + \lambda_3\lambda_5 + \lambda_3\lambda_6 + \lambda_4\lambda_5 + \lambda_4\lambda_6 + \lambda_5\lambda_6$,
3. $p_3 = \lambda_1\lambda_2\lambda_3 + \cdots + \lambda_4\lambda_5\lambda_6$,
4. $p_4 = \lambda_1\lambda_2\lambda_3\lambda_4 + \cdots + \lambda_3\lambda_4\lambda_5\lambda_6$,
5. $p_5 = \lambda_1\lambda_2\lambda_3\lambda_4\lambda_5 + \lambda_1\lambda_2\lambda_3\lambda_4\lambda_6 + \lambda_1\lambda_2\lambda_3\lambda_5\lambda_6 + \lambda_1\lambda_2\lambda_4\lambda_5\lambda_6 + \lambda_1\lambda_3\lambda_4\lambda_5\lambda_6 + \lambda_2\lambda_3\lambda_4\lambda_5\lambda_6$, and
6. $p_6 = \lambda_1\lambda_2\lambda_3\lambda_4\lambda_5\lambda_6$.

Notice that $p_1$ is the trace of $A$, $p_6$ is the determinant of $A$, and the characteristic polynomial equation of $A$ is $\lambda^6 - p_1\lambda^5 + p_2\lambda^4 - p_3\lambda^3 + p_4\lambda^2 - p_5\lambda + p_6 = 0$.

Recall Newton's identities for symmetric functions [11], state

$$s_1 - p_1 = 0,$$

$$s_2 - p_1 s_1 + 2p_2 = 0,$$

$$s_3 - p_1 s_2 + p_2 s_1 - 3p_3 = 0,$$

$$s_4 - p_1 s_3 + p_2 s_2 - p_3 s_1 + p_4 = 0,$$

$$s_5 - p_1 s_4 + p_2 s_3 - p_3 s_2 + p_4 s_1 - p_5 = 0,$$

$$s_6 - p_1 s_5 + p_2 s_4 - p_3 s_3 + p_4 s_2 - p_5 s_1 + p_6 = 0.$$

Now solving for the coefficients in terms of the moments when the trace of $A$ is zero, that is $p_1 = 0$, we find:

$$\begin{aligned}
p_1 &= s_1 = 0 \\
p_2 &= -\frac{1}{2}s_2 \\
p_3 &= \frac{1}{3}s_3 \\
p_4 &= \frac{2p_2^2 - s_4}{4} \\
&= \frac{\frac{s_2^2}{2} - s_4}{4} \\
&= \frac{s_2^2 - 2s_4}{8} \\
p_5 &= \frac{-p_3 s_2 + p_2 a_3 + s_5}{5} \\
&= \frac{-\frac{s_2 s_3}{3} - \frac{s_2 s_3}{2} + s_5}{5} \\
p_6 &= \frac{p_3 s_3 - s_6 - p_2 s_4 - p_4 s_2}{6} \\
&= \frac{\frac{s_3}{3}s_3 - s_6 + \frac{s_2}{2}s_4 - \frac{s_2^2 - 2s_4}{8}s_2}{6} \\
&= \frac{8s_3^2 - 3s_2^3 - 24s_6 + 18s_2 s_4}{144}
\end{aligned}$$

Using the above with the knowledge that $p_6$ is the determinant of $A$ we have the following special case of the JLL condition.

**Theorem 33.** *Let $A$ be a singular, trace zero, $6 \times 6$ matrix with spectrum $\sigma = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6\}$. Then $3s_6 \geqslant s_3^2$.*

*Proof.* As $A$ is singular we know $p_6 = 0$. Therefore,

$$-24s_6 + 18s_2s_4 + 8s_3^2 - 3s_2^3 = 0$$

$$3s_2\left(6s_4 - s_2^2\right) = 24s_6 - 8s_3^2$$

but $s_2 \geqslant 0$ and the JLL inequality gives $6s_4 - {s_2}^2 \geqslant 0$, so

$$24s_6 - 8s_3^2 \geqslant 0$$

$$3s_6 \geqslant s_3^2.$$

□

## 7.3 Conclusion

This chapter analyzes the possible outcomes for lists of six real numbers in order to solve the SNIEP and the RNIEP, and resolves 65 out of a possible 79 cases for the SNIEP.

We now turn to the final chapter of new results regarding the NIEP where we have added a twist to the problem. Here we address whether or not a list of $n$ numbers can be the spectrum of a nonnegative matrix that is subordinate to a bipartite graph.

# CHAPTER 8

# NIEP for Matrices Subordinate to a Given Graph

## 8.1 Graph Version of the NIEP

Although most of this dissertation has addressed the NIEP in a general setting mathematically it is natural to consider a graph version of the NIEP. Recall a *graph*, $G$, is a set of vertices, $\{v_1, v_2, \ldots, v_n\}$ and a set of edges, $\{e_1, e_2, \ldots, e_m\}$ that connect vertices. We will consider simple graphs with no loops or multiple edges. A *directed graph* is one in which there is a direction associated with each edge. The graph theoretic version of the NIEP requires that if two vertices do not have an edge between them in a given graph $G$, then the matrix realizing $\sigma = \{\lambda_1, \ldots, \lambda_n\}$ must have a zero entry in that position. So for a directed graph where the edge has direction from $v_k$ to $v_s$, the realizing matrix $A = (a_{ij})$ must have a zero entry in $a_{sk}$. If the graph is undirected, then a realizing matrix $A = (a_{ij})$ must have zeroes located in the $a_{ks}$ and $a_{sk}$ entries when $v_k$ and $v_s$ are not joined by an edge. We say an $n$ x $n$ matrix $A = (a_{ij})$ is *subordinate* to a graph $G$ if $a_{ij} \neq 0$ for $i \neq j$ implies that there is an edge in $G$ connecting vertex $v_i$ with $v_j$. Note that the diagonal entries are unrestricted.

*The Graph Nonnegative Inverse Eigenvalue Problem (G-NIEP) relative to a graph $G$ on $n$ vertices asks to identify the necessary and sufficient conditions for lists of $n$*

*complex numbers to occur as the eigenvalues of a nonnegative matrix subordinate to the given graph G.* Similarly we define the G-RNIEP and the G-SNIEP. Notice that in the G-SNIEP the graph $G$ must be undirected and the list of $n$ numbers must be real as the realizing matrices are symmetric. Leal-Duarte and Johnson point out in their paper [19] that requiring the realizing matrix to be subordinate to $G$ rather than having the graph of the realizing matrix be exactly $G$, guarantees that the solution set to each problem is always a closed set.

When we select the type of graph we open a gamut of new problems. In our case we will restrict ourselves to the G-NIEP when the graph in question is a bipartite graph. A *bipartite graph* is an undirected graph $G$ in which the vertices may be partitioned into two parts in such a way that all edges have a vertex in each part. A *complete bipartite graph* is a bipartite graph having all allowed edges. Associated with each (bipartite) graph is a matching number denoted $m = m(G)$. The *matching number* of any graph is the maximum number of vertex disjoint edges of the graph. In the case of a bipartite graph, $m(G)$ is less than or equal to the smaller number of the vertices in its parts. In the case where $G$ is a complete bipartite graph, $m(G)$ will equal the smaller cardinality of the two parts.

Some answers are already known regarding these problems. Leal-Duarte and Johnson [19] resolved the G-SNIEP for matrices subordinate to a bipartite graph. We will discuss their result in the next section.

## 8.2 G-SNIEP for Matrices Subordinate to a Bipartite Graph

In the paper by Leal-Duarte and Johnson [19], we find that simple linear inequalities are all that are necessary and sufficient to describe a solution to the G-SNIEP for matrices subordinate to a bipartite graph. Their solution for $\sigma = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$ where $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$ and $\lambda_1 + \lambda_n \geqslant 0$, is:

$$\left\{\begin{array}{c} \lambda_1 + \lambda_n \geqslant 0 \\ \lambda_2 + \lambda_{n-1} \geqslant 0 \\ \vdots \\ \lambda_m + \lambda_{n-m+1} \geqslant 0 \\ \lambda_{m+1} \geqslant 0 \\ \vdots \\ \lambda_{n-m} \geqslant 0 \end{array}\right\} \tag{23}$$

where $m = m(G)$. If $n$ is even, where $m(G) = \frac{1}{2}n$, the lower single entry inequalities do not occur. The following lemmas walk us through the proof given by Leal-Duarte and Johnson that (23) gives a complete solution to the G-SNIEP for matrices subordinate to a bipartite graph. The first lemma shows the necessity of these inequalities, while the second lemma shows the sufficiency for the existence of a symmetric nonnegative matrix subordinate to $G$.

**Lemma 2.** [19] *Let $G$ be a bipartite graph on $n$ vertices with matching number $m$ and suppose that $A$ is a symmetric nonnegative matrix subordinate to $G$. The eigenvalues of $A$ satisfy the inequalities of* (23).

*Proof.* Let $A = D + B$ where $D$ is a nonnegative diagonal matrix and $B$ is a nonnegative symmetric matrix with diagonal entries all zero. Then rank $(B) \leqslant 2m\,(G)$. This is because no submatrix whose size is larger than $2m\,(G)$ can have a nonzero term in its determinant. Also as $B$ is similar to $-B$ via the signature matrix with 1's in the diagonal entries corresponding to one of the parts of $G$ and -1's in the other diagonal entries, we have

$$\begin{gathered}\mu_n = -\mu_1 \\ \mu_{n-1} = -\mu_2 \\ \vdots \\ \mu_{n-m+1} = -\mu_m \\ \mu_{m+1} = 0 \\ \vdots \\ \mu_{n-m} = 0\end{gathered}$$

in which $\mu_1 \geqslant \cdots \geqslant \mu_n$ are the eigenvalues of $B$. But, since $D$ is positive semidefinite, we have, using the Weyl inequalities [11], that

$$\lambda_i \geqslant \mu_i, \text{ for } i = 1, \ldots, n$$

in which $\lambda_1 \geqslant \cdots \geqslant \lambda_n$ are the eigenvalues of $A$. It follows that the $\lambda_i$'s satisfy the inequalities (23). $\square$

**Lemma 3.** [19] *Let $G$ be a bipartite graph on $n$ vertices with matching number $m$ and suppose that $\lambda_1 \geqslant \cdots \geqslant \lambda_n$ satisfy the inequalities of* (23). *Then there exists a nonnegative symmetric $n$ x $n$ matrix subordinate to $G$ with eigenvalues $\lambda_1, \ldots, \lambda_n$.*

*Proof.* As we have seen in Chapter 5, for any two real numbers $\alpha$ and $\beta$ such that $\alpha + \beta \geqslant 0$ there is a 2 x 2 symmetric nonnegative matrix having eigenvalues $\alpha$ and $\beta$. Now, consider a collection of edges of $G$ that realize $m(G)$. Suppose, without loss of generality, that they are $\{1,2\}$, $\{3,4\}$, ..., $\{2m-1,2m\}$. Construct a 2 x 2 symmetric nonnegative matrix with eigenvalues $\lambda_1, \lambda_n$ and call it $A_1$, and repeat this process constructing a 2 x 2 symmetric nonnegative matrix with eigenvalues $\lambda_k, \lambda_{n-k+1}$ and calling it $A_k$ for $k = 2, \ldots, m$. Further let the 1 x 1 matrix $A_j = [\lambda_j]$, for $j = m+1, \ldots, n-m$. Now $A_1 \oplus \cdots \oplus A_m \oplus A_{m+1} \oplus \cdots \oplus A_{n-m}$ is a symmetric nonnegative $n$ x $n$ matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ that is subordinate to $G$. $\square$

Now combining the two lemmas together, Leal-Duarte and Johnson give the simple yet elegant result.

**Theorem 34.** [19] *Let $G$ be a bipartite graph on $n$ vertices with matching number $m$. There is a symmetric nonnegative $n$ x $n$ matrix subordinate to $G$ and with eigenvalues $\lambda_1, \ldots, \lambda_n$ if and only if $\lambda_1, \ldots, \lambda_n$ satisfy the inequalities* (23).

We state a few more results derived by Leal-Duarte and Johnson from Theorem 34 and the notion of independence number of a graph. The *independence number*, denoted $i(G)$, of an undirected graph $G$ is the maximum number of vertices of $G$ among which there are no edges. In the case of a bipartite graph $G$, $i(G) = n - m(G)$. The first result gives an upper bound on the number of negative and nonnegative eigenvalues a symmetric nonnegative matrix subordinate to a bipartite graph can have. The second result puts a lower bound on the number of nonnegative eigenvalues

a symmetric nonnegative matrix subordinate to an undirected graph can have.

**Corollary 5.** [19] *An $n$ x $n$ symmetric nonnegative matrix subordinate to the bipartite graph $G$, has at most $m(G)$ negative eigenvalues and at least $n - m(G)$ nonnegative eigenvalues.*

**Theorem 35.** [19] *An $n$ x $n$ symmetric nonnegative matrix subordinate to the undirected graph $G$ has at least $i(G)$ nonnegative eigenvalues.*

Since this section wraps up the solution to the G-SNIEP for matrices subordinate to a bipartite graph and does so with relative simplicity, we consider next the G-NIEP problem for a bipartite graph $G$. We'll see in the next section that although the G-NIEP is a natural extension, a solution may not have the same simplicity and elegance that the G-SNIEP solution does.

## 8.3 G-NIEP for Matrices Subordinate to a Bipartite Graph

Let us consider a list of complex numbers $\sigma = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ where $\lambda_1 \geqslant |\lambda_i|$ for all $i = 2, \ldots, n$. We'll try to find necessary and sufficient conditions on such a list to solve the G-NIEP for trace zero nonnegative matrices subordinate to a bipartite graph $G$. We already know certain necessary conditions on $\sigma$ namely: $\sigma$ is closed under complex conjugation, the Perron root must be an element of $\sigma$, the moments must be nonnegative, and the JLL condition must hold. In addition to these, the following theorem gives two more necessary conditions for a spectrum of a non-symmetric, nonnegative, trace zero matrix subordinate to a bipartite graph.

**Theorem 36.** *Let $A$ be a non-symmetric, nonnegative, trace zero $n$ x $n$ matrix subordinate to a bipartite graph. Then $\sigma(A) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ must satisfy the following:*

1. *$\sigma(A)$ is closed under multiplication by -1 and*

2. *$f$ $n$ is odd, then $0 \in \sigma(A)$.*

*Proof.* Let $\sigma(A) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ be the spectrum of a non-symmetric, nonnegative, trace zero $n$ x $n$ matrix, $A$, subordinate to a bipartite graph. Then $\sigma$ is a subset of the complex numbers and must contain the Perron root because $A$ is nonnegative. We also have that $\sum_{i=1}^{n} \lambda_i = 0$ by the trace zero assumption.

As $\sigma$ is the set of roots of the characteristic equation for $A$ which has real coefficients, we know it must be closed under complex conjugation.

We may consider $A$ to be permutation similar to the nonnegative matrix having the form

$$\begin{bmatrix} 0_{11} & A_{12} \\ A_{21} & 0_{22} \end{bmatrix} \tag{24}$$

where $0_{11}$ is the $k$ x $k$ zero matrix and $0_{22}$ is the $(n-k)$ x $(n-k)$ zero matrix. This is because $A$ is subordinate to a bipartite graph where we are assuming the two vertex disjoint subsets have cardinalities $k$ and $n-k$. So without loss of generality let $A$ be of this form. Then by Theorem 6, we know that the eigenvalues of this matrix are the square roots of the eigenvalues of the product $A_{12}A_{21}$ which is a $k$ x $k$ matrix. This means we have at most $2k$ nonzero eigenvalues where the remaining $n-2k$ eigenvalues must be zero for $k < \frac{n}{2}$. Now since the product $A_{12}A_{21}$ is a nonnegative

matrix, we have the spectrum of $A_{12}A_{21}$ to be closed under complex conjugation. So for $re^{i\theta} \in \sigma(A_{12}A_{21})$, it follows that $re^{-i\theta} \in \sigma(A_{12}A_{21})$. Therefore the square roots of these two numbers will be in $\sigma(A)$. That is, $\pm\sqrt{r}e^{\frac{i\theta}{2}} \in \sigma(A)$ and $\pm\sqrt{r}e^{\frac{-i\theta}{2}} \in \sigma(A)$. As this is at most four eigenvalues and contains the additive inverse of each one, we find that $\sigma(A)$ is closed under multiplication by -1.

Now due to the restriction that $\sigma(A)$ is closed under multiplication by -1, we find that the natural pairing of $\lambda_i \in \sigma(A)$ with $-\lambda_i \in \sigma(A)$ can be done. So if $n$ is odd, $trace(A) = 0$ requires that the remaining unpaired eigenvalues must be zero. Thus $0 \in \sigma(A)$. □

The next result allows us to construct a matrix having $\sigma = \{\pm\sqrt{\lambda_1}, \ldots, \pm\sqrt{\lambda_n}\}$ when we know of a nonnegative matrix having $\lambda_1, \ldots, \lambda_n$, as its eigenvalues.

**Theorem 37.** *Let $A$ be a nonnegative $n$ x $n$ matrix having eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then the matrix $B$ defined by*

$$B = \begin{bmatrix} 0 & A \\ I & 0 \end{bmatrix},$$

*where $I$ is the $n$ x $n$ identity matrix, is a nonnegative, trace zero, 2n x 2n matrix subordinate to a bipartite graph with matching number $n$. Moreover, the eigenvalues of $B$ are $\pm\sqrt{\lambda_1}, \pm\sqrt{\lambda_2}, \ldots, \pm\sqrt{\lambda_n}$.*

*Proof.* Using Theorem 6, let $A_{21} = I$ and $A_{12} = A$ where each one is an $n$ x $n$ matrix . Then the eigenvalues of $A_{21}A_{12} = IA = A$ are indeed the eigenvalues of $A$. Then we find by Theorem 6, that the eigenvalues of $B$ are indeed $\pm\sqrt{\lambda_1}, \pm\sqrt{\lambda_2}, \ldots, \pm\sqrt{\lambda_n}$. □

Theorem 36 and Corollary 37 are used for solving the G-NIEP subordinate to a bipartite graph in the case where $\sigma = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ contains seven or fewer elements. We can construct solutions to these cases using the results from prior chapters.

Let us consider the following cases for the G-NIEP for trace zero matrices subordinate to a bipartite graph. Let $\lambda_1$ be the Perron root in all cases.

1. In the case where $\sigma = \{\lambda_1\}$, the only solution is when $\lambda_1 = 0$. Then such a nonnegative, trace zero matrix is the zero matrix, and it is subordinate to a bipartite graph with one vertex and no edges.

2. Consider the case where $\sigma = \{\lambda_1, \lambda_2\}$. First of all, to satisfy Theorem 36, $\lambda_2 = -\lambda_1$. This means $\lambda_2$ is real since $\lambda_1$ is. The solution to such a set $\sigma = \{\lambda_1, -\lambda_1\}$ is:
$$A_2 = \begin{bmatrix} 0 & \lambda_1^2 \\ 1 & 0 \end{bmatrix}.$$
This matrix has eigenvalues $\pm\lambda_1$ and is subordinate to a bipartite graph on two vertices.

3. Consider the case where $\sigma = \{\lambda_1, \lambda_2, \lambda_3\}$. As we know, since $\lambda_1 \in \sigma$ it follows that $-\lambda_1 \in \sigma$ too. Without loss of generality, let $\lambda_2 = -\lambda_1$. Since the trace is

zero, we have $\lambda_3 = 0$. Thus using the result from (2), we have

$$A_3 = \begin{bmatrix} 0 & \lambda_1^2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

This is a nonnegative matrix subordinate to a bipartite graph having three vertices with matching number 1.

4. Consider the case where $\sigma = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$. We may assume $\lambda_2 = -\lambda_1$ and $\lambda_4 = -\lambda_3$. Note that $\lambda_3$ could be a pure imaginary number or a real number. Since $\lambda_3^2$ could be negative, a necessary requirement for a 2 x 2 matrix with eigenvalues $\lambda_1^2$, $\lambda_3^2$ to exist is that $\lambda_1^2 + \lambda_3^2 \geqslant 0$. But this requirement follows from the necessary condition on $\sigma$ that $\lambda_1 \geqslant |\lambda_3|$.

   Therefore using

$$B = \frac{1}{2}\begin{bmatrix} \lambda_1^2 - \lambda_3^2 & \lambda_1^2 + \lambda_3^2 \\ \lambda_1^2 + \lambda_3^2 & \lambda_1^2 - \lambda_3^2 \end{bmatrix},$$

   we find a nonnegative matrix subordinate to a bipartite graph on four vertices with matching number 2 using Theorem 37, namely

$$A_4 = \begin{bmatrix} 0 & B \\ I & 0 \end{bmatrix}.$$

5. Consider the case where $\sigma = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\}$. Using Theorem 36 we know that one of the values in $\sigma$ must be 0, so assume $\lambda_5 = 0$. Then the remaining

numbers must satisfy the conditions from (4). Let $A_4$ be the nonnegative matrix having eigenvalues $\lambda_1, \lambda_2, \lambda_3, \lambda_4$. Then the following nonnegative matrix is subordinate to a bipartite graph on five vertices with matching number 2 and has eigenvalues $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$.

$$A_5 = \begin{bmatrix} A_4 & 0 \\ 0 & 0 \end{bmatrix}$$

6. Consider the case where $\sigma = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6\}$. This is the first time a complex number with nonzero real part and nonzero imaginary part, may be an element of $\sigma$. This is because $\sigma$ must be closed under complex conjugation and additive inverse, and have the Perron root as an element. For notation purposes, let $\lambda_i$ and $\lambda_{i+1}$ for $i$ odd, be the additive inverses of each other.

Therefore we can use the 3 x 3 symmetric nonnegative solution from Theorem 9 to construct the nonnegative matrix $A$ having the eigenvalues $\lambda_1^2, \lambda_3^2, \lambda_5^2$. Then by Theorem 37, we have the nonnegative matrix subordinate to a bipartite graph on six vertices with matching number 3. For example,

$$B = \begin{bmatrix} 0 & 0 & 0 & 2 & 3 & 4 \\ 0 & 0 & 0 & 6 & 7 & 8 \\ 0 & 0 & 0 & 4 & 6 & 7 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

is a matrix subordinate to the complete bipartite graph $K_{3,3}$ and

$$\sigma(B) = \{\pm 4.03997, \pm(0.408884 - 0.572597i), \pm(0.408884 + 0.572597i)\}.$$

7. Consider the case where $\sigma = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7\}$. By Theorem 36 as seven is odd, we know that one of the eigenvalues of $\sigma$ must be zero. So we may use the construction technique of (6) to create a nonnegative matrix, $B$, having spectrum $\sigma - \{0\}$. Then our solution for a nonnegative matrix subordinate to a bipartite graph on seven vertices with matching number 3 is

$$\begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix}.$$

### 8.4 Conclusion

While this chapter gives a partial solution to the G-NIEP for bipartite graphs, it becomes apparent that continuing on in this fashion requires the solution to the NIEP in the general setting.

## BIBLIOGRAPHY


[1] Berman, A. and Plemmons, R. J. *Nonnegative Matrices in the Mathematical Sciences.* Academic Press: New York, 1979.

[2] Borobia, A. On the nonnegative eigenvalue problem. *Linear Algebra and Its Applications* 223-224, (1995). 131-140.

[3] Boyle, M. Handelman, D., Kim, H. and Roush, F. The spectra of nonnegative matrices via symbolic dynamics. *Annals of Mathematics* 133, (1991). 249-316.

[4] Ciarlet, P. G. Some results in the theory of nonnegative matrices. *Linear Algebra and Its Applications* 1, (1968). 139-152.

[5] Dmitriev, N. and Dynkin, E. On the characteristic numbers of a stochastic matrix. *Dokl. Nauk SSSR* 49, (1945). 159-162.

[6] Dmitriev, N. and Dynkin, E. Eigenvalues of a stochastic matrix. *Izv. Akad. Nauk SSSR Ser. Mat.* 10, (1946). 167-184.

[7] Fiedler, M. Eigenvalues of nonnegative symmetric matrices. *Linear Algebra and Its Applications* 9, (1974). 119-142.

[8] Friedland, S. On an inverse problem for nonnegative and eventually nonnegative matrices. *Israel Journal of Mathematics 29* 1, (1978). 44-60.

[9] Frobenius G. Über matrizen aus nicht negativen elementen. *S.-B. Preuss. Akad. Wiss. (Berlin)*, (1912). 456-477.

[10] Hershkowitz, D. *Existence of Matrices Satisfying Prescribed Conditions.* M. Sc. thesis, Technion-Israel Institute of Technology, 1978.

[11] Horn, R. A. and Johnson, C. R. *Matrix Analysis.* Cambridge University Press, 1985.

[12] Johnson, C. Row stochastic matrices similar to doubly stochastic matrices. *Linear and Multilinear Algebra* 10, (1981). 113-130.

[13] Johnson, C., Laffey, T. J. and Loewy, R. The real and the symmetric nonnegative inverse eigenvalue problems are different. *Proceedings of the American Mathematical Society* 124, (1996). 3647-3651.

[14] Karpelevich, F. On the eigenvalues of a matrix with nonnegative elements. *Izv. Akad. Nauk SSSR Ser. Mat.* 15, (1951). 361-383.

[15] Kellogg, R. B. Matrices similar to a positive or essentially positive matrix. *Linear Algebra and Its Applications* 4, (1971). 191-204.

[16] Knudsen, C. and McDonald J. J. A note on the convexity of the realizable set of eigenvalues for nonnegative symmetric matrices. *The Electronic Journal of Linear Algebra* 8, (2001). 110-114.

[17] Laffey, T. and Meehan, E. A refinement of an inequality of Johnson, Loewy, and London on nonnegative matrices and some applications. *The Electronic Journal of Linear Algebra* 3, (1998). International Linear Algebra Society. 119-128.

[18] Laffey, T. and Meehan, E. A characterization of trace zero nonnegative 5 x 5 matrices. *Linear Algebra and Its Applications* 302-303, (1999). 295-302.

[19] Leal-Duarte, A. and Johnson, C. R. Resolution of the symmetric nonnegative inverse eigenvalue problem for matrices subordinate to a bipartite graph. To be published in *Positivity*, 2001, preprint.

[20] Loewy, R. and London, D. A note on an inverse problem for nonnegative matrices. *Linear and Multilinear Algebra* 6, (1978). 83-90.

[21] McDonald, J. J. and Neumann, M. The Soules approach to the inverse eigenvalue problem for nonnegative symmetric matrices of order $n \leqslant 5$. *Contemporary Mathematics* 259, (2000). 387-407.

[22] Minc, *Nonnegative Matrices.* John Wiley and Sons: New York, 1988.

[23] Perfect, H. On positive stochastic matrices with real characteristic roots. *Duke Mathematical Journal* 48, (1952). 271-276.

[24] Perfect, H. Methods of constructing certain stochastic matrices. *Duke Mathematical Journal* 20, (1953). 395-404.

[25] Perfect, H. Methods of constructing certain stochastic matrices. II *Duke Mathematical Journal* 22, (1954). 305-311.

[26] Perron, O. Zur theorie der über matrizen. *Math. Ann.* 64, (1907). 248-263.

[27] Radwan, N. An inverse eigenvalue problem for symmetric and normal matrices. *Linear Algebra and Its Applications* 248, (1996). 101-109.

[28] Reams, R. *Topics in Matrix Theory,* Thesis presented for the degree of Ph.D., National University of Ireland, Dublin, 1994.

[29] Reams, R. An inequallity for nonnegative matrices and the inverse eigenvalue problem. *Linear and Multilinear Algebra* 41, (1996). 367-375.

[30] Salzmann, F. L. A note on eigenvalues of nonnegative matrices. *Linear Algebra and Its Applications* 5, (1972). 329-338.

[31] Soules, G. W. Constructing symmetric nonnegative matrices. *Linear and Multilinear Algebra* 13, (1983). 241-251.

[32] Suleimanova, K. R. Stochastic matrices with real eigenvalues (in Russian). *Soviet Math. Dokl.* 66, (1949). 343-345.

[33] van der Waerden, B. L. *Modern Algebra 1.* Frederick Ungar. New York, 1988.

[34] Wuwen, G. Eigenvalues of nonnegative matrices. *Linear Algebra and Its Applications* 266, (1997). 261-270.